AN OPTIMAL ALGORITHM FOR EXPERIMENTAL DESIGN IN INVERSION WITH APPLICATION TO SUBSURFACE SENSING

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ABSTRACT

We consider the optimal placement of multiple transmitter and receiver antennas for subsurface sensing of soil heterogeneities. The technique employed is based on the theory of optimal experiments, and may be used in general inverse-scattering problems. Specifically, the method represents the inversion as a parameter-estimation problem, where the parameters of interest are defined by the heterogeneity (target) profile. Each possible antenna placement constitutes a class of data that may be measured (defining an “experiment”). The algorithm is based on the Fisher information matrix, and the antennas are placed in positions to maximize the Fisher matrix. Example results are presented for the problem of imaging the spatial electrical properties of layered materials (e.g., soils), here based on simulated data.

Key words: Inversion, Optimization, Fisher Matrix.

1. INTRODUCTION

There have been numerous techniques developed for general inverse-scattering problems [1-6], with many of these approaches appropriate for subsurface-sensing applications [7-10]. In most of these previous studies it has been assumed that the sensor characteristics are fixed or predetermined, and the focus has been on developing appropriate inverse techniques given the data available. However, for many problems optimal design of the underlying measurements is a significant challenge, motivating integration of the sensing and inversion tasks, such that the measured data is well matched to the inversion goals, leading to improved inversion quality.

The idea of integrating the sensing and inversion components has been gaining significant attention recently. For example, researchers have investigated the optimal positions of cameras, with the goal of inverting for the shape of a target viewed by optical sensors [11]. Similar techniques have been considered for sensing of subsurface targets using an induction sensor on a robotic platform [12]. These general adaptive formalisms have also been used in general signal-processing settings [13,14]. In addition, we note that integrating the sensing and processing tasks is also an important component of general robotic sensing problems, where investigators have studied such techniques as Markov decision processes and partially observed Markov decision processes [15].

The work presented here represents an extension of some of the above ideas to the specific problem of wave-based inverse scattering, where here we consider electromagnetic inversion for the electrical parameters of a subsurface heterogeneity. The techniques considered fall under the general rubric of optimal experiment design [16]. This theory has been developed over several decades in the statistics community [16-18], and is focused on designing experiments that produce informative data for a given objective. In the subsurface-sensing problem of interest here, we wish to invert for the electrical contrast of a particular heterogeneity, with respect to a background medium [8]. The error between the underlying model and measured fields is characterized as a Gaussian process, with this error manifested by fundamental limitations of the underlying model as well as possible additive Gaussian noise in the data itself. We design an experiment (here corresponding to antenna placement) by using the Fisher information matrix [19].

In the work presented here we demonstrate how the Fisher information matrix may be formulated and approximated using the underlying wave equation, for a general class of subsurface-sensing problems. Current-loop antennas are considered, and the objective is to determine the optimal antenna position and orientation for the inversion task. Although the technique is general, results are presented for the specific problem of characterizing the layered electrical properties of soils, based on a set of subsurface antennas.

As demonstrated, with particular assumptions design of the optimal antenna positions may be performed without first explicitly performing the inversion or collecting data. However, to demonstrate the accuracy of the inversion based on the specified antenna design, we also show inversion results, based on computed data with additive Gaussian noise with iterative extended-Born method [8].

The remainder of the paper is organized as follows. In Sec. 2 we briefly introduce the inverse setting of interest here, defining the specific problem under consideration. The method used to design the inversion measurements is detailed in Sec. 3. We demonstrate that with particular
assumptions this task becomes relatively straightforward, avoiding the need for any measured data when performing experiment design. Example results are presented in Sec. 4, and conclusions are discussed in Sec. 5.

2. INVERSION PROBLEM

2.1 Problem Statement

The theory of optimal experiments [16] is applicable to many inverse-scattering problems; we here consider subsurface sensing problem that is summarized in Fig. 1. A heterogeneity is assumed embedded within a lossy dielectric half space. Antennas may be placed on the four sides of a cube, which is assumed to completely surround the heterogeneity. For $M$ transmitting antennas and $M$ receiving antennas, a total of $M^2$ multistatic measurements are assumed, with these performed at a fixed frequency. The theory may also be used to determine the optimal set of frequencies to use within the inversion, but that is not considered here. Our objective is to place the $M$ sensors (coil antennas) at optimal positions and orientations.

Figure 1: Schematic of subsurface-sensing and inversion problem.

2.2 Forward Model

In the inversion we require data from the various sensor positions (see Fig. 1). It is from these (in practice measured) fields that the inversion is performed. In this study these fields are computed via a reference forward model, employing a volumetric formulation, solved via the method of moments (MoM) [8] with the dyadic half-space Green's function computed via the complex-image technique (CIT) [8].

In all forward computations, the currents on the excitation loop are assumed known and unchanged. Hence, the incident electric field, due to a loop current of amplitude $I$ is calculated as

$$E^{inc}(r) = \int \frac{1}{C} G_{El}(r,r') \cdot dI'$$  

where $C$ is the circle of the loop and $G_{El}(r,r')$ is the dyadic half-space Green's function representing induced electric fields in terms of excitation electric currents.

In the iterative inversion process, we require the electric fields inside the heterogeneity, due to the known source; this is complicated by the fact that these fields are a function of the heterogeneity, which is unknown and to be determined. Therefore, at each iteration of the inversion, the electric fields inside the domain are computed using the heterogeneity profile from the previous iteration. Since these electric fields must be computed numerous times, the forward solver employed in the inversion must be accurate and efficient. In our study we use a form of the extended-Born method [8] as the forward solver. Specifically, assume that $E(r)$ represents the electric field at position $r$ and that $E^{inc}(r)$ denotes the incident electric fields produced by the excitation. We therefore have

$$E(r) = E^{inc}(r) + j \omega \varepsilon_{r} \int \left[ \varepsilon_{b}(r) - \varepsilon_{r}(r') \right] G_{El}(r,r') \cdot E(r') d^3r'$$  

where $\varepsilon_{b}$ represents the generally inhomogeneous dielectric constant of the background, and $\varepsilon_{r}$ represents the dielectric constant of the heterogeneity which is assumed to occupy the volume $V$. In general the half-space background medium and the heterogeneity in domain $V$ are lossy, and therefore both $\varepsilon_{b}$ and $\varepsilon_{r}$ are complex.

Let $V = \bigcup_{n=1}^{N} V_{n}$ with $V_{n}$ the $n$th small mesh cubic volume, then (2) may be expressed as

$$E(r) = E^{inc}(r) + \sum_{n=1}^{N} j \omega \varepsilon_{r} \int \left[ \varepsilon_{b}(r) - \varepsilon_{r}(r') \right] G_{El}(r,r') \cdot E(r') d^3r'$$

We may approximate

$$E(r_{k}) = E^{inc}(r_{k}) + j \omega \varepsilon_{r} \sum_{n=1}^{N} \int \left[ \varepsilon_{b}(r_{k}) - \varepsilon_{r}(r_{k}) \right] G_{El}(r_{k},r_{k}') \cdot E^{inc}(r_{k}) d^3r'$$

$$+ \sum_{n=1}^{N} j \omega \varepsilon_{r} \int \left[ \varepsilon_{b}(r_{k}) - \varepsilon_{r}(r_{k}) \right] G_{El}(r_{k},r_{k}') \cdot E^{inc}(r_{k}) d^3r'$$

$$k = 1, 2, \ldots, N$$

where $r_{k}$ is the center point of $V_{n}$. In the inversion algorithm, the forward model employed is derived from (4)

$$E(r_{k}) = M(r_{k}) \left[ E^{inc}(r_{k}) + j \omega \varepsilon_{r} \sum_{n=1}^{N} G_{El}(r_{k},r_{k}') E^{inc}(r_{k}') \right]$$

$$M(r_{k}) = I - j \omega \varepsilon_{r} \sum_{n=1}^{N} G_{El}(r_{k},r_{k}')$$

$$G_{El}(r_{k},r_{k}') = \int_{V_{n}} G_{El}(r_{k},r_{k}') d^3r'$$

where $I$ is a $3 \times 3$ unit matrix and $a_{n} = [\varepsilon_{b}(r_{k}) - \varepsilon_{r}(r_{k})] \varepsilon_{r}(r_{k})$.

2.3 Inversion Model

We wish to invert for the contrast $a(r)$ based on the measured voltage at the small receiver loops. Assuming that we have computed the field inside the target domain via the method in (5), we have the equivalent current

$$J(r) = j \alpha \varepsilon_{r} E(r) - \varepsilon_{r}(r) E(r)$$

Then the voltage in receiver loop at $r_{m}$

$$V_{i}(r_{m}) = - \frac{d}{dt} B_{i} \cdot n_{m} \cdot dS$$

$$= \Delta \mu_{s} E_{i} \Delta \varepsilon_{m} \int_{V} a(r') G_{El}(r_{m},r') \cdot E_{i}(r') d^3r'$$

where $E_{i}$ and $V_{i}$ are respectively the electric field and voltage induced by the $i$th transmitter loop, $\Delta S$ is the
(small) area of the loop, \( n_m \) is the normal of the receiver loop at \( r_m \), and \( G_{H}(r, r') \) is the dyadic half-space Green’s function [8] representing induced magnetic fields in terms of excitation electric currents.

From (7) the numerical method for inversion may be expressed in terms of the following linear equations

\[
\sigma^2 \mu_e \varepsilon_0 \Delta S \sum_{n=1}^{N} [n_m \cdot G_{H}(r_n, r_{m}) \cdot E_{i}(r_{m})] r_{m} = V_{i}(r_{m}) \tag{8}
\]

where \( G_{H}(r_{m}, r') = \int G_{H}(r_{m}, r') d^3 r' \). There are a total of \( M \times M \) equations with \( N \) unknowns \( a = (a_1, a_2, \ldots, a_N)^{\top} \), represented in matrix form as \( \mathbf{Z}a = \mathbf{v} \).

As is well known for inversion problems of this type, the matrix equation \( \mathbf{Z}a = \mathbf{v} \) is over-determined when \( M \times M > N \) and underdetermined when \( M \times M < N \). In principle one may solve such equations via the corresponding least-squares analysis \( \min \| \mathbf{Z}a - \mathbf{v} \|^{2} \).

However, typically such an inversion is ill-posed, and therefore we have found it necessary to introduce regularization. In this research we use a simple Tikhonov regularization, i.e., we seek \( a \) that minimizes \( \| \mathbf{Z}a - \mathbf{v} \|^{2} + \beta\| a \|^{2} \), where \( \beta \) is a regularization parameter [8]. As demonstrated in Section 4 when presenting example results, this regularization is often effective. The complete algorithm for the inversion problem consists of repetition of (5) and (8).

### 3. OPTIMAL ALGORITHM FOR LOOP POSITIONS

#### 3.1 Statistical Model

To generate Fisher information matrix with which the optimal algorithm for loop positions will be developed, we must establish statistical models for the observed data first.

Considering the Born approximation \( E(r) = E_{inc}(r) \) for \( r \) within the heterogeneity, from (7) we have

\[
V_{i}(r_{m}) = \sigma^2 \mu_e \varepsilon_0 \Delta S n_m \cdot \int \alpha(r) G_{H}(r_n, r_{m}) \cdot E_{inc}(r_{m}) d^3 r'
\]

where \( E_{inc} \) is the incident field generated by the \( l \)th transmitter loop, and \( V_{i}(r_{m}) \) is the associated measured voltage at the \( m \)th receiver. Let \( U_{i}(r_{m}) = V_{i}(r_{m}) / (\sigma^2 \mu_e \varepsilon_0 \Delta S) \) and

\[
U_{i}(r_{m}) = \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) + \mathcal{U}_{i}(r_{m}), \quad l, m = 1, 2, \ldots, M \tag{9}
\]

where \( \mathcal{U}_{i}(r_{m}) \) represents the error manifested by the approximations employed. The error may be caused by fundamental approximations in the wave equations (for noise-free measured data), and for the presence of additive noise in the measured data (if present).

We now define a \( 2M^2 \) -dimensional vector

\[
U = (\text{Re}(U_{1}(r_{1})), \ldots, \text{Re}(U_{1}(r_{m})), \text{Im}(U_{1}(r_{1})), \ldots, \text{Im}(U_{1}(r_{m})), \text{Re}(U_{2}(r_{1})), \ldots, \text{Re}(U_{2}(r_{m})), \text{Im}(U_{2}(r_{1})), \ldots, \text{Im}(U_{2}(r_{m})))^{\top}
\]

and \( 2M \times 2M \) matrix

\[
a = (a_{1}, a_{2}, \ldots, a_{N})^{\top} = (\text{Re}(a_{1}), \text{Re}(a_{2}), \ldots, \text{Re}(a_{N}), \text{Im}(a_{1}), \ldots, \text{Im}(a_{N}))^{\top}
\]

Further, assume for simplicity that the errors \( \text{Re}(\partial \mathcal{U}_{1}(r_{m})) \), \( \text{Im}(\partial \mathcal{U}_{1}(r_{m})) \), \( l, m = 1, 2, \ldots, M \) are zero-mean Gaussian with variance \( \sigma^2 \). Thus

\[
p(\text{Re}(U_{i}(r_{m}))) = N \left( \text{Re}\left( \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \right), \sigma \right)
\]

\[
p(\text{Im}(U_{i}(r_{m}))) = N \left( \text{Im}\left( \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \right), \sigma \right)
\]

which also may be expressed as

\[
p(U | a) = \frac{1}{(2\pi\sigma^2)^{M^2}} \prod_{i=1}^{M} \prod_{m=1}^{M} \text{exp} \left[ -\frac{1}{2\sigma^2} \left| U_{i}(r_{m}) - \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \right|^{2} \right]
\]

\[
= \frac{1}{(2\pi\sigma^2)^{2MN}} \prod_{i=1}^{M} \prod_{m=1}^{M} \text{exp} \left[ -\frac{1}{2\sigma^2} \left| U_{i}(r_{m}) - \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \right|^{2} \right]
\]

#### 3.2 Fisher Information Matrix

By the definition, the \((i,j)\) entry \( H_{ij} \) of Fisher information matrix \( H \) is

\[
H_{ij} = -E \left[ \frac{\partial^2 \ln p(U | a)}{\partial a_{i} \partial a_{j}} \right] = \int \frac{\partial^2 \ln p(U | a)}{\partial a_{i} \partial a_{j}} p(U | a) dU
\]

Let \( g_{i,n} = n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \)

and

\[
Q = \sum_{i=1}^{M} \sum_{m=1}^{M} \left| U_{i}(r_{m}) - \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \right|^{2}
\]

then

\[
\frac{\partial^2 \ln p(a | U)}{\partial a_{i} \partial a_{j}} = -\frac{1}{2\sigma^2} \frac{\partial^2 Q}{\partial a_{i} \partial a_{j}}
\]

By (16) and (17), we have

\[
\frac{\partial^2 Q}{\partial a_{i} \partial a_{j}} = \sum_{i=1}^{M} \sum_{m=1}^{M} \left| U_{i}(r_{m}) - \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \right|^{2}
\]

\[
= \sum_{i=1}^{M} \sum_{m=1}^{M} \left| U_{i}(r_{m}) - \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \right|^{2}
\]

\[
= \sum_{i=1}^{M} \sum_{m=1}^{M} \left| U_{i}(r_{m}) - \sum_{l=1}^{L} a_{l} n_{m} \cdot G_{H}(r_{m}, r_{n}) \cdot E_{inc}(r_{n}) \right|^{2}
\]
\[ Q = \sum_{j=1}^{M} \sum_{i=1}^{N} U_j(r_m) - \sum_{n=1}^{N} \alpha_n g''_{mn} \]
\[ = \sum_{j=1}^{M} \sum_{i=1}^{N} \left( \text{Re} \left[ U_j(r_m) - \sum_{n=1}^{N} \alpha_n g''_{mn} \right] \right)^2 + \sum_{j=1}^{M} \sum_{i=1}^{N} \left( \text{Im} \left[ U_j(r_m) - \sum_{n=1}^{N} \alpha_n g''_{mn} \right] \right)^2 \]
\[ = \sum_{j=1}^{M} \sum_{i=1}^{N} \left( \text{Re} \left[ U_j(r_m) - \sum_{n=1}^{N} \alpha_n g''_{mn} \right] \right)^2 + \sum_{j=1}^{M} \sum_{i=1}^{N} \left( \text{Im} \left[ U_j(r_m) - \sum_{n=1}^{N} \alpha_n g''_{mn} \right] \right)^2 \]

(19)

hence
\[ \frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j} = \begin{cases} 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \left[ \text{Re} \left( g_{mn}^l \right) \text{Re} \left( g_{mn}^l \right) + \text{Im} \left( g_{mn}^l \right) \text{Im} \left( g_{mn}^l \right) \right], & \text{if } i, j \leq N \\ 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \left[ \text{Im} \left( g_{mn}^l \right) \text{Re} \left( g_{mn}^l \right) - \text{Re} \left( g_{mn}^l \right) \text{Im} \left( g_{mn}^l \right) \right], & \text{if } i \leq N, j > N \\ 2 \sum_{m=1}^{M} \sum_{n=1}^{N} \left[ \text{Re} \left( g_{mn}^l \right) \text{Re} \left( g_{mn}^l \right) + \text{Im} \left( g_{mn}^l \right) \text{Im} \left( g_{mn}^l \right) \right], & \text{if } i, j > N \end{cases} \]

Defining the 2N x N matrix
\[
G_m^l = \begin{pmatrix} 
\text{Re} \left( g_{mn}^l \right) & \text{Im} \left( g_{mn}^l \right) \\
\text{Re} \left( g_{mn}^l \right) & \text{Im} \left( g_{mn}^l \right) \\
\vdots & \vdots \\
\text{Re} \left( g_{mn}^l \right) & \text{Im} \left( g_{mn}^l \right) \\
\text{Im} \left( g_{mn}^l \right) & \text{Re} \left( g_{mn}^l \right) \\
\vdots & \vdots \\
-\text{Im} \left( g_{mn}^l \right) & \text{Re} \left( g_{mn}^l \right) \\
\vdots & \vdots \\
\end{pmatrix}, \quad l, m = 1, 2, \ldots, M 
\]

(21)

then the matrix with entries \[ \frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j} \] may be expressed as
\[
\frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j} = 2 \sum_{m=1}^{M} \sum_{n=1}^{N} G_m^l G_m^l \]  
(22)

since \[ g_{mn}^l \] is independent of \[ U \], by (15), (18) and (22) we obtain
\[
H_y = \frac{1}{2 \sigma^2} E \left[ \frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j} \right] = \frac{1}{2 \sigma^2} \frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j} 
\]

(23)

Finally by (22) and (23) we get Fisher information matrix
\[
H = \frac{1}{\sigma^2} \sum_{i=1}^{M} G_m^l G_m^l \]  
(24)

3.3 Optimal Experiment Design

The expression of Fisher information matrix in (24) is defined in terms of the matrices \[ G_m^l \], which are a function of the background medium and the domain to be imaged – is independent of the specific heterogeneity considered. The particular antenna positions for the \[ M^2 \] multi-static measurements are manifested in the double sum in (24). Our objective is to position the \[ M \] antennas.

There are many measures one may employ in terms of \[ H \], to characterize the quality of the \( M \) antennas [16]. We here compute the trace of \( H \). Our objective is to choose the positions of transmitter and receiver loops to maximize the trace of the Fisher information matrix \( H \) [19]; this yields a final optimization procedure, as discussed below.

Assume that \( M - 1 \) antennas have been employed, and we wish to determine where the \( M \)th antenna should be placed. We may quantify the utility of the \( M - 1 \) antennas as
\[
\text{trace}(H_{M-1}) = \text{trace} \left[ \sum_{i=1}^{M-1} G_i^l G_i^l \right] 
\]

We choose the \( M \)th antenna position and orientation to maximize
\[
\text{trace}(H_M) = \text{trace} \left[ \sum_{i=1}^{M} G_i^l G_i^l \right] 
\]

with the first \( M - 1 \) loop positions and orientations fixed. Since
\[
\text{trace}(H_M) = \text{trace}(H_{M-1}) + \text{trace} \left[ \sum_{m=1}^{M-1} G_m^l \left( G_m^l \right)^T \right] + \text{trace} \left[ \sum_{i=1}^{M-1} G_i^l \left( G_i^l \right)^T \right] + \text{trace} \left[ G_m^l \left( G_m^l \right)^T \right] 
\]

(25)

We repeat the above step until the number of loop positions reaches the desired number or until the increment in (25) becomes sufficiently small. We note that, after much analysis, the actual experimental design reflected in (25) is relatively simple.

There are two important implications of the above analysis, which motivated key assumptions made at the start. First, note that the Fisher information matrix \( H \) is independent of the particular measured scattered fields \( U \). Secondly, (17) is quadratic in the heterogeneity profile \( a \), and therefore the \( H \) is independent of the particular heterogeneity profile considered (see (25)); it is only dependent on the domain to be imaged and on the associated background medium. Both of these results, which make experimental design relatively simple, are a direct consequence of the Born approximation used in (9). We note, however, that in the final inversion (Sec. 2) the Born approximation is not required – it is only used here to place the antennas optimally, within the aforementioned approximations.

4. EXAMPLE RESULTS

4.1 Problem Statement

We consider characterization of the subsurface complex dielectric constant of model soil, based on measurements performed using buried loop antennas. Although a detailed exposition on the application is not the main focus of the paper, we note that this study is
motivated by the goal of monitoring the near-surface electrical properties of soil, as they evolve due to weather and other environmental changes. For this application it is assumed that small loop antennas will be placed in the ground, and used to regularly monitor the evolution of the subsurface electrical properties. The presence of the small loops is assumed to provide a minor perturbation to the subsurface electrical parameters (e.g., weakly altering the evolution of soil characteristics).

The problem is modeled using a half-space background Green’s function, here defined by relative dielectric parameters \( \varepsilon_r = 5 - 0.2j \) at frequency 100 MHz. The domain to be imaged is 1.44 m x 1.44 m x 1.44 m, and includes the soil just below the air-soil interface. In the inversion and in the experimental design the cubic meshes have a width of 9 cm; we note that this is distinct from the mesh used to compute the “measured” scattered fields for the subsequent inversion (12 cm), motivated by the desire of avoiding an “inverse crime” [3].

As indicated in Fig. 1, the loop antennas in this study may be placed on the four sides of the domain to be imaged, and as discussed further below each of these four surfaces is partitioned into 11 x 11 possible antenna positions along each of the four sides (with inter-element spacing 14.4 cm), for a total of 440 possible antenna positions (loops on the edges are shared by two sides). In addition, each of these coil antennas may be oriented with their axis in the \( x, y \) or \( z \) direction (see Fig. 2), for a total of 1320 possible antenna positions and orientations. The four planes on which the antennas may be placed are situated on the sides of the 1.44 m x 1.44 m x 1.44 m domain to be imaged.

4.2 Optimal Antenna Placement

We present here optimal antenna placement using the algorithm discussed in Sec. 3. In this example we consider \( M=60 \) total antennas, and we started with \( M=4 \) initial antenna positions and orientations, with these indicated in red in Fig. 3. In Fig. 3, at each point for which there is no antenna a “0” is displayed, and the three numbers indicate antenna dipoles situated in the following directions: “1” represents the magnetic dipole directed normal to plane and into the domain, “2” represents the moment oriented in the \( z \) direction, and “3” represents the moment directed parallel to the planar surface in either the “\( x \)” or “\( y \)” direction (depending on the plane in question). Note the symmetry that is naturally preserved for sides 1 and 3, and sides 2 and 4. We also note that the algorithm doesn’t select a dipole oriented with moment normal to the sides.

With regard to selection of \( M=60 \), in this example the information content continues to increase by adding new

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Figure 3: Antenna locations and orientations, where “1” indicates a moment normal to the side of the inversion domain, “2” denotes a dipole moment in the \( z \) direction, and “3” denotes a dipole in the \( y \) (Sides 2 or 4) or \( x \) (Sides 1 or 3) directions, see Fig. 2. A “0” indicates no antenna is used at the corresponding point.
antennas. This indicates that the information content is anticipated to continue to improve for \( M > 60 \). We have stopped the algorithm at \( M = 60 \) since we desired a relatively sparse antenna array (such that it doesn't disturb the soil properties, as indicated above). Therefore, these results may be interpreted as representing the optimal antenna placements (within the approximations noted in Sec. 3) for a prescribed number of \( M = 60 \) antennas. A similar optimal design may be synthesized for any \( M \) of interest.

**4.3 Inversion Results**

Using the antenna design denoted in Fig. 3, we now perform an inversion using the iterative algorithm [8] summarized in Sec. 2. We again emphasize that the measured data for this study are computed for the antenna array in Fig. 3, for all \( M^2 \) transmit-receive pairs, using a rigorous volumetric forward solver [8]. Within the inversion algorithm we employ a distinct and approximate extended-Born forward solver [8].

In this example the “measured” data are computed for a layered media, with variation in the \( z \) direction (the layered medium is embedded within the half-space background discussed above). Specifically, the true layered media is described by

\[ \varepsilon_{\text{\small true}} = 5.0 - 0.2 j, \quad z \in [-1.44, -1.08] \]
\[ \varepsilon_{\varepsilon} = 5.6 - 0.1 j, \quad z \in [-1.08, -0.72] \]

where \( z = 0 \) corresponds to the air-soil interface, and the above units are in meters. In the top layer we note that there is a contrast of approximately 40% relative to the background half space, with this reducing to zero contrast as one proceeds to the bottom layer. We also note that within the inversion discussed in Sec. 2, the Tikhonov regularization parameter (see Sec. 2.3) is set as \( \beta = 5 \times 10^{-6} \).

Zero-mean white Gaussian noise is added independently to the real and imaginary parts of the “measured” signal at all sensor positions, with the noise standard deviation 10% of the average root-mean-square (rms) signal. Inversion results are shown in Fig. 4, for the sequence of vertical slices. Each of these vertical “slices” is composed of 256 cubes, each of width 9 cm. We note that the slices in the interior of the domain, away from the antennas, resolve the heterogeneity profile well (we plot the magnitude of the estimated contrast at each point). The “truth” for this inversion, given the parameters discussed above, is represented as \( |a| = 0.37, \quad |a| = 0.24, \quad |a| = 0.11 \) and \( |a| = 0.0 \), from top to bottom (for the four layers indicated above). These results represent convergence after two iterations of the iterative inversion algorithm discussed in Sec. 2.
We attribute the relatively poor inversion results near the antennas to two phenomena. First, the fields vary quickly in the vicinity of the antennas, with this complexity possibly undermining inversion quality. In addition, there is an abrupt discontinuity between the half-space background and the layered media characteristic of the heterogeneity, and the antennas reside right at this discontinuity. This latter phenomenon is expected to diminish as the antennas are placed further inside the domain to be imaged.

To make the inversion quality quantitative, in Fig. 5 we plot the relative error between the true and estimated profile, defined as

$$E = \frac{\sum_{i=1}^{N} |a_i(r_i) - a_{inv}(r_i)|}{\sum_{i=1}^{N} |a_i(r_i)|}$$

where $a_i(r_i)$ represents the true contrast at position $r_i$ and $a_{inv}(r_i)$ represents the associated inverted profile. The results in Fig. 5 are plotted as a function of percentage of additive noise, as defined above, with actual data computed in increments of 5% noise levels.

![Figure 4](image1.png)  
![Figure 5](image2.png)

**Figure 4:** Inversion results, vertical slices 1-16, for a layered medium, using the antenna design reflected in Fig. 3.

**Figure 5:** Relative error between the true and estimated heterogeneity profile, as a function of percentage additive white Gaussian noise.

### 5. Discussion and Conclusions

An algorithm has been developed based on the theory of optimal experiments [16], with the goal of defining the optimal antenna positions for performing an inversion based on measured fields. We have considered low-frequency (100 MHz) electromagnetic fields, but the same framework is applicable as well to acoustic scattering data. We have demonstrated that by employing the Born approximation, the optimal design of antennas may be determined prior to collecting any
data. We have considered this case in our example results. However, the basic design framework may be extended beyond the Born approximation, but in this case the design must be performed iteratively as new data are collected (i.e., measured data are required and determined adaptively). We note that after the experimental setup has been constituted, any (e.g., non-Born) inversion method may be employed. Consequently, the use of the Born approximation within the design does not imply that the actual contrast/size of the problem has to be weak/small (as required for the Born approximation [4]), but the final design may not be rigorously optimal if this is not the case.

Example results have been presented for the problem of monitoring the near-surface electrical properties of soils, with this problem motivating the design framework presented here. Results of the antenna design have been presented, in addition to corresponding results based on an iterative Born inversion [8]. In future work one may consider other parameters to vary when performing the experimental design. For example, one may also consider investigating the optimal choice of sensor frequencies. Moreover, one may wish to choose between multiple antenna types.

REFERENCES