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SIGNIFICANCE OF COVARIANT DERIVATIVE IN FORMULATING GAUGE TRANSFORMATIONS

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ABSTRACT

The invariance of the laws of physics under local symmetries predict the existence of a covariant derivative. The various forms of interactions present in the nature appear as a natural consequence of this invariance.
Keywords- *Abelian Symmetry, Covariance, QED, SU(3), Gauge Theory*

1. INTRODUCTION

This brief discussion aims to complement the available study material on Gauge Theories and provide a short insight into what gauge theories are, how they work and why we use them.Historically, while trying to explain the quantum effects of electrodynamics, it was found QED can be explained by a U(1) abelian gauge theory. Yang and Mills [1,2] then generalised this abelianU(1) gauge theory to the non-abelian gauge theory case. It is shown that gauge transformations leave the action and the classical equations of motion invariant. The phase of the wave-function itself is unobservable in quantum mechanics. Only the phase differences are observable e.g. via interference phenomena. The phase difference is visible in the Aharonov–Bohm effect [3].

2. GAUGE FIELDS

The states of gauge fields in Quantum Electrodynamics (QED)are members of $U(1)_{EM}$ group Therefore the number of gauge fields is equal to $dim(U(1)_{EM})$ or 1. This gauge field is the photon.It couples to charged leptons and quarks. It is known that Spontaneous Symmetry Breaking creates a mass term in the Lagrangian and so is responsible for generating mass. However SSB does not occur in QED and So the photons remain massless.

The states of gauge fields in Quantum Chromodynamics (QCD) are members of $SU(3)_{colour}$.group. QCD offers a new way of thinkingabout matter. Every quark field of flavour f, say $\Psi^f(x)$, has an associated color of red, green or blue. Let us define the complete wave function

$$\chi^f(x) = \begin{pmatrix} \Psi^f_{red}(x) \\ \Psi^f_{green}(x) \\ \Psi^f_{blue}(x) \end{pmatrix}$$

The gauge invariant Lagrangian can be constructed, of the form

$$\mathcal{L} = \bar{\chi}^f (i\gamma^\mu D_\mu - m)\chi^f$$

This Lagrangian is invariant under $SU(3)_{colour}$, i.e. $\chi^f(x) \to U(x)\chi^f(x)$where $U(x) \in SU(3)_{colour}$. The number of gauge fields is equal to the dimensions of $SU(3)_{colour}$ (or $dim(SU(3)_{colour})$ which is 8. These are the eight gluon fields that couple to the quarks, holding them together to formnon-pertubative bound states called hadrons. Spontaneous symmetry breaking does not occur in QCD and so the gluons remain massless.

The states of gauge fields in Elecroweaktheory are members of the product of gauge groups $SU(2)_{left}\times U(1)_{hypercharge}$. The number of gauge fields is equal to $dim(SU(2)_{left}\times U(1)_{hypercharge}) = 4$. These gauge fields are the $W^a_\mu, B_\mu,\ a = 1,2,3$. Spontaneous symmetry breakingdoes occur in Electroweak theory. The gauge group of Electroweak theory i.e$SU(2)_{left}\times U(1)_{hypercharge}$interacts with a complex doublet. A complex doublethas four real scalar field components. The symmetry group is then spontaneously broken to $U(1)_{QED}$. Each broken generatorproduces a goldstone boson. The number of goldstone bosons is, therefore, equal to the number of broken generators or lost dimensions i.e.$dim(G/H) = dim(SU(2)_{left}\times U(1)_{hypercharge}) - dim(U(1)_{QED}) = 4- 1 = 3$. These three goldstone bosons are “eaten” by three gauge fields to become massive (in unitarygauge). These massive gauge fields are called $W^\pm_\mu, Z^0_\mu$. The $W^\pm_\mu$ couple to left handed matter causing flavour changing processes like beta decay, Z^0_μ couples to all matter particles. The massless gauge field is called the photon which couples to charged matter only. We have one massive real scalar field left after SSB, which we call the Higgs.

3. INVARIANCE OF LAGRANGIAN

A symmetry is a transformation of the fields that leaves the action (and hence “physics”) invariant. A global symmetry is a symmetry that does not depend on space-time. Global symmetries give rise to conserved currents and charges as described by Noether’s theorem. On the other hand, gauge symmetry is a continuous local symmetry where the symmetry group is continuous and depends on space-time. Gauge symmetries introduce gauge fields to the theory which mediate a force.

An example of a global symmetry is the phase transformation is

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$$\Psi(x) \rightarrow e^{i\alpha}\Psi(x); \overline{\Psi}(x) \rightarrow e^{-i\alpha}\overline{\Psi}(x) \qquad (1)$$

where α is a parameter which does not depend on spacetime, x. We know, a free fermion field is described by the Dirac Lagrangian

$$\mathcal{L} = \overline{\Psi}(x)(i\gamma^{\mu}\partial_{\mu} - m)\Psi(x) \qquad (2)$$

Using transformation laws for $\Psi(x)$ and $\overline{\Psi}(x)$ as given in Eq.(1). Lagrangian $\mathcal{L}$ will transform to

$$\mathcal{L} \rightarrow \overline{\Psi}(x)e^{-i\alpha}(i\gamma^{\mu}\partial_{\mu} - m)e^{i\alpha}\Psi(x) = \mathcal{L} \qquad (3)$$

Since α does not depend on space-time, x, we can commute it past the derivative and so Lagrangian $\mathcal{L}$remains invariant.

A local symmetry is a symmetry that depends on space-time, x. A gauge symmetry is a local symmetry where the symmetry group is continuous e.g. U(N), SU(N).

Let's say we have an N component column vector of fermion fields with identical masses,

$$\chi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \vdots \\ \Psi_N(x) \end{pmatrix} \qquad (4)$$

Then, just as in the previous example, the Lagrangian

$$\mathcal{L} = \bar{\chi}(x)\left[(i\gamma^{\mu}\partial_{\mu} - m)I_{N\times N}\right]\chi(x) \qquad (5)$$

will be invariant under a global U(N) transformation given by

$$\chi(x) \rightarrow \Omega\chi(x) \quad \text{where} \quad \Omega \in U(N)$$
$$\bar{\chi}(x) \rightarrow \bar{\chi}(x)\Omega^{\dagger} \qquad (6)$$

The transformation Ω preserves local symmetry, that means it will be a function of space and time or $\Omega \rightarrow \Omega(x)$. How does the Lagrangian transform now?

$$\mathcal{L} = \bar{\chi}(x)\Omega^{\dagger}(x)\left[i\gamma^{\mu}\partial_{\mu} - m\right]\Omega(x)\chi(x)$$
$$\mathcal{L} = \bar{\chi}(x)\Omega^{\dagger}(x)\left[i\gamma^{\mu}\partial_{\mu}(\Omega(x)\chi(x)) - m\Omega(x)\chi(x)\right]$$
$$\mathcal{L} = \bar{\chi}(x)\Omega^{\dagger}(x)\left[i\gamma^{\mu}\Omega(x)\partial_{\mu}\chi(x) + i\gamma^{\mu}\partial_{\mu}(\Omega(x))\chi(x) - m\Omega(x)\chi(x)\right]$$
$$= \mathcal{L} + \bar{\chi}(x)\Omega^{\dagger}(x)i\gamma^{\mu}\partial_{\mu}(\Omega(x))\chi(x) \qquad (7)$$

Which is not invariant because we cannot commute the transformation matrix $\Omega(x)$ past the derivative because it is a space-time function. Therefore the "physics" is invariant under the transformation

$$\chi(x) \rightarrow \Omega(x)\chi(x) \qquad (8)$$

The problem is that derivative ∂_{μ}does not transform covariantly.

$$\partial_{\mu}(\chi(x)) \nrightarrow \Omega(x)\,\partial_{\mu}\chi(x)$$

However, we have to construct a covariant derivative, say D_{μ} which will transform covariantly in order to make our Lagrangian invariant under the transformations. Therefore

$$D_{\mu}(\chi(x)) \rightarrow \Omega(x)D_{\mu}\chi(x) \qquad (9)$$

Let us consider the ordinary derivative in some direction n^{μ}:

$$\partial_{\mu}\chi(x) = \lim_{\epsilon\rightarrow 0}\frac{1}{\epsilon}\left(\chi(x+\epsilon) - \chi(x)\right) \qquad (10)$$

where ϵ is a small displacement in n^{μ} direction.

The problem arises because $\chi(x+\epsilon)$ and $\chi(x)$ do not transform identically because of space-time dependence of the transformation $\Omega(x)$.

Therefore $\chi(x+\epsilon)$ transforms to $\Omega(x+\epsilon)\chi(x+\epsilon)$ and not to $\Omega(x)\chi(x+\epsilon)$ whereas $\chi(x)$ is still transforming to $\Omega(x)\chi(x)$. The two fields at different spacetime points cannot be compared because of change in value of the co-ordinates 'x' themselves.Therefore, the derivative ∂_{μ}, as defined in Eq.(10) loses its normal meaning.We need to construct a new derivative with the two field values $\chi(x)$and $\chi(x+\epsilon)$ transforming identically.

Let us define transformation $U(y, x) \in SU(N)$, called parallel transport, which itself transforms under a gauge transformation as

$$U(y,x) \rightarrow \Omega(y)U(y,x)\Omega^{\dagger}(x) \qquad (11)$$

Therefore

$$U(y,x)\chi(x) \rightarrow \Omega(y)U(y,x)\Omega^{\dagger}(x)\Omega(x)\chi(x)$$
$$= \Omega(y)U(y,x)\chi(x) \qquad (12)$$

This shows $U(y,x)\chi(x)$ transforms like $\chi(y)$ and as such is also well defined. This is exactly what we need for the derivative to work. Now let us choose $y = x + \epsilon$ and define the new covariant derivative as

$$D_{\mu}\chi(x) = \lim_{\epsilon\rightarrow 0}\frac{1}{\epsilon}\left[\chi(x+\epsilon) - U(x+\epsilon, x)\chi(x)\right] \qquad (13)$$

Each term in this equation, by construction transforms in the same way under the gauge transformation, so that now

$$D_{\mu}(\chi(x)) \rightarrow \Omega(x)D_{\mu}\chi(x)$$

Therefore the covariant derivative transforms covariantly.The problem is now to find an explicit form for D_{μ} for a Lagrangian which will be invariant under local symmetries. As we know, the parallel transport $U(y,x)$ is given by [4]

$$U(y,x) = \exp\left(ig\int_{\Gamma} A_{\mu}(x)dx^{\mu}\right)$$

Where Γ is a path joining y to x. (14)

Since $U(y,x) \in SU(N)$ and therefore, $A_{\mu}(x)$ is an element of the Lie algebra of SU(N) with coupling g.

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Choosing y= x + ε, we get for infinitesimal path Γ from x to y = x + ε

$$U(x+\varepsilon,x) \approx \exp(ig\epsilon A_\mu(x)) \approx 1 + ig\epsilon A_\mu(x) + O(\epsilon^2) \quad (15)$$

Substituting for U(x + ∈, x) into Eq.(13), we get

$$D_\mu\chi(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[\chi(x+\epsilon) - U(x+\epsilon,x)\chi(x)]$$

$$= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}[\chi(x+\epsilon)-\chi(x)] + igA_\mu(x)\chi(x) + O(\epsilon)\chi(x)\right) = [\partial_\mu - igA_\mu(x)]\,\chi(x) \quad (16)$$

Therefore, the covariant derivative $D_\mu = \partial_\mu - igA_\mu(x)$ where $A_\mu(x)$ is an element of the Lie algebra of SU(N). The transformation properties of $A_\mu(x)$ can be found as follows

Substituting $y = x + \varepsilon$ in Eq. (11), we get

$$U(x+\varepsilon,x) \to \Omega(x+\varepsilon)U(x+\varepsilon,x)\Omega^\dagger(x)$$

But from Eq.(14), we have

$$U(x+\varepsilon,x) \approx \exp\left(ig\epsilon A_\mu(x)\right) \approx 1 + ig\epsilon A_\mu(x) + O(\epsilon^2)$$

Therefore

$$1 + ig\epsilon A_\mu(x) + O(\epsilon^2) \to \Omega(x+\varepsilon)[1 + ig\epsilon A_\mu(x) + O(\epsilon^2)]\Omega^\dagger(x)$$

Using expansion

$$\Omega(x+\varepsilon) = \Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)$$

$$1 + ig\epsilon A_\mu(x) + O(\epsilon^2) \to \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right)[1 + ig\epsilon A_\mu(x) + O(\epsilon^2)]\Omega^\dagger(x)$$

$$= \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right)\Omega^\dagger(x) + \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right) ig\epsilon A_\mu(x)\Omega^\dagger(x) + \left(\Omega(x) + \varepsilon\,\partial_\mu\Omega(x) + O(\epsilon^2)\right) O(\epsilon^2)\Omega^\dagger(x)$$

$$= 1 + \varepsilon\,\partial_\mu\Omega(x)\Omega^\dagger(x) + \Omega(x)\, ig\epsilon A_\mu(x)\Omega^\dagger(x) + \varepsilon\,\partial_\mu\Omega(x)\, ig\epsilon A_\mu(x)\Omega^\dagger(x) + O(\epsilon^2)$$

Neglecting $O(\epsilon^2)$ on either side and cancelling 1, we get

$$ig\epsilon A_\mu(x) \to \epsilon\,\partial_\mu\Omega(x)\Omega^\dagger(x)(1 + \varepsilon\, igA_\mu(x)) + ig\epsilon\Omega(x)A_\mu(x)\Omega^\dagger(x)$$

Dividing on either side by$ig\epsilon$ and taking limit $\epsilon \to 0$, we get on rearranging

$$A_\mu(x) \to \Omega(x)A_\mu(x)\Omega^\dagger(x) - \frac{i}{g}\partial_\mu\Omega(x)\Omega^\dagger(x) \quad (17)$$

which is the required transformation equation for $A_\mu(x)$. It makes the derivative $D_\mu = \partial_\mu - igA_\mu(x)$ transform covariantly

$$D_\mu\chi(x) \longrightarrow \Omega(x)D_\mu\chi(x)$$

Finally we have a Lagrangian

$$\mathcal{L} = \bar{\chi}(x)\left[\left(i\gamma^\mu D_\mu - m\right)I_{N\times N}\right]\chi(x) \quad (18)$$

Which is now invariant under the local transformations of the fields

4. COVARIANT DERIVATIVE D_μ OBEYS LEIBNITZ RULE

Let Ψ_1, Ψ_2 be two fields and g_1, g_2 be corresponding couplings and then we should have

$$D_\mu(\psi_1\psi_2) = (\partial_\mu - i(g_1+g_2)A_\mu)(\psi_1\psi_2)$$
$$D_\mu\psi_1 = (\partial_\mu - ig_1A_\mu)\psi_1$$
$$D_\mu\psi_2 = (\partial_\mu - ig_2A_\mu)\psi_2 \quad (19)$$

Then $(D_\mu\psi_1)\psi_2 + \psi_1(D_\mu\psi_2)$

$$= \left((\partial_\mu - ig_1A_\mu)\psi_1\right)\psi_2 + \psi_1\left((\partial_\mu - ig_2A_\mu)\psi_2\right)$$
$$= \left((\partial_\mu\psi_1)\psi_2 - (ig_1A_\mu)\psi_1\psi_2\right) + \left(\psi_1(\partial_\mu\psi_2) - \psi_1(ig_2A_\mu)\psi_2\right)$$
$$= (\partial_\mu\psi_1)\psi_2 + \psi_1(\partial_\mu\psi_2) - iA_\mu(g_1+g_2)\psi_1\psi_2$$
$$= \partial_\mu(\psi_1\psi_2) - iA_\mu(g_1+g_2)\psi_1\psi_2$$
$$= D_\mu(\psi_1\psi_2) \qquad \text{from Eq.(19)}$$

This shows that a covariant derivative D_μ just as ordinary derivative obeys the Leibnitz rule.

$$D_\mu(\psi_1\psi_2) = (D_\mu\psi_1)\psi_2 + \psi_1(D_\mu\psi_2) \quad (20)$$

5. CONCLUSIONS

Gauge transformations are the local transformations and are functions of space and time. The invariance of physical laws requires the quest for a covariant derivative as the ordinary derivative loses its meaning under gauge transformations. The covariant derivative differs from the ordinary derivative by a term responsible for the interactions. Gauge theories thus, develop as a natural consequence of the invariance of Lagrangian under gauge transformations.

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