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Study on a Particular Form of Cubic Equation

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Abstract- In this paper the general form of cubic equation has been modified into a particular form and has been thoroughly studied. To evaluate exact roots of the modified form, a simplified formula has been derived and presented. This derived formula is similar to that of Sridharacharya's formula for solving quadratic equation and the similarities has been pointed out in this paper. Furthermore this particular form behaves in a special manner when Cardan's method is applied on it for finding the roots. In the preliminaries section basic definitions, rules for round off and the Cardan's method for solving cubic equations have been stated in this paper for better understanding purposes. Apart from finding out the roots of the special types of cubic equation, the nature of the roots has also been discussed here. To illustrate utility of the derived formula appropriate examples have been provided. A brief historical background of evaluation of cubic equation is also given in the introduction section and names of various well known methods to solve the cubic equations are also mentioned here. This derived formula will be very helpful for solving the particular types of cubic equations discussed here. More over the derived formula is very easy to remember and recall in the time of need. Some interesting observations have been made throughout the paper. The similarity of the derived formula with that of Sridharacharya's formula is matter of interest in the paper.

Keywords- Polynomial equation; Depressed Cubic equation; Sridharacharya's formula; Real root; Imaginary root; Cardan's method

I. INTRODUCTION

The history of evaluation of the cubic equations is a fascinating one in itself. It is belived that ancient Babylonians, Indians, Greeks, Chinese and Egyptians were familiar with the cubic equations. The Babylonian Cuneifer tablets (20th-16th century BC) has been found with tables for calculating cubes and cube roots. But it is not known if the tables could have been used by the Babylonian to solve cubic equations or not. Throughout the time many mathematicians have come up with various varients of cubic equations and their solutions. The major breakthrough came with the publication of a general method of solving cubic equations by Geronimus Cardano in his Ars Magna(1545). Tartaglia had also claimed to find the solution to the reduced cubic equations in 1535 but didn't published it. Discussion about the history of the evaluation of cubic equation is not the aim of this paper and hence not going into the historical journey. Till date numerious methods have evolved for finding roots of cubic equations. Some of the methods are:

- Cardan's method of solving cubic equation(Algebraic method).
- Trigonometric method (Francois Viete (1540-1603) owns the credit for independently finding the trigonometric solutions of cubic equations with three real roots)
- Numerical method (eg. Method of bisection, Newton's method)

- The cubic polynomial equations can also be solved using funtional evalution.
- In this paper a particular form of cubic equation has been studied throughly.

II. PRELIMINARIES

Definition

Polynomial

Let $a_0, a_1, \dots, a_n$ be real numbers, then $\sum_{i=0}^{n} a_i x^i$ is a polynomial in x(with real coefficients). When $a_0, a_1, \dots, a_n$ are not all zero, it can be assumed that $a_n \neq 0$ and the polynomial has degree n.

The number a_r is the coefficient of x^r (for $= 0,1,2,\dots,n$).

Polynomial_Equation

An equation $f(x) = 0$, where $f(x)$ is a polynomial is called a polynomial equation.

Cubic_Equation

A polynomial equation of degree three is called a cubic equation.

A general form of cubic equation is given as $ax^3 + bx^2 + cx + d = 0$ with $a \neq 0$.

Depressed_Cubic_Equation

A cubic equation of the form $tx^3 + rx + m = 0$ with $t \neq 0$ is called a depressed cubic equation.

This are much simplier than the general cubic equation and since by suitable substitution any cubic can be transformed to depressed cubic equation so they are of much importance.

Cardan's_Method_for_Solving_Cubic_Equations

General form of cubic equation is $ax^3 + bx^2 + cx + d = 0$ with $a \neq 0$.

Let $x = y + r$,where r is such a quantity that coefficient of y^2 will be zero in the transformed equation.

After substitution of the value of x the transformed equation will be of the form

$$y^3 + 3Uy + V = 0 \quad (2.1)$$

Let y=h+g

$$\therefore y^3 = h^3 + g^3 + 3hg(h+g) = h^3 + g^3 + 3hgy$$
$$\therefore\ y^3 - 3hgy - (h^3 + g^3) = 0 \qquad (2.2)$$

Equation (2.1) and (2.2) being identical we have $U = -hg$ i.e., $h^3 g^3 = -U^3$ and $V = -(h^3 + g^3)$ i.e.,

$$h^3 + g^3 = -V$$

Now h^3 and g^3 are the roots of the quadratic equation

$$k^2 + Vk - U^3 = 0 \qquad (2.3)$$

Let $m^3, n^3$are the roots of equation (2.3), which are h^3, g^3 respectively.

$\therefore h = m, m\omega, m\omega^2$ and $g = n, n\omega, n\omega^2$ where $\omega(= -\frac{1}{2} + \frac{\sqrt{3}}{2} i)$ and $\omega^2(= -\frac{1}{2} - \frac{\sqrt{3}}{2} i)$ are imaginary cube root of unity.

So $y = m + n, m\omega + nw^2, m\omega^2 + n\omega$

Hence $x = m + n - r, m\omega + nw^2 - r, m\omega^2 + n\omega - r$

Rules_For_Round_Off

To round off a number upto n decimal places discard all the numbers right to the n-th place and the number in the n-th place will be

- unchanged if the number in the (n+1)-th place is any one of the values 0,1,2,3,4
- increased by 1 if the number in the (n+1)-th place is any one of the values 6,7,8,9
- incresed by 1 if the (n+1)-th number is 5 and atleast one nonzero number is following it.
- increased by 1 if the (n+1)-th number is 5 and no other nonzero number is following it with an odd number in the n-th place.
- unchanged if the (n+1)-th number is 5 and no other nonzero number is following it with an even number in the n-th place.

III. MAIN RESULTS

1. General Discussion and Finding Roots

The general form of cubic equation is given as $ax^3 + bx^2 + cx + d = 0$ with $a \neq 0$. If $c = \frac{b^2}{3a}$, then the above equation becomes

$$ax^3 + bx^2 + \frac{b^2}{3a}x + d = 0 \qquad (3.1)$$
$$\Rightarrow 3a^2x^3 + 3abx^2 + b^2x + 3ad = 0$$
$$\Rightarrow 9a(3a^2x^3) + 9a(3abx^2) + 9ab^2x + 9a(3ad) = 0$$
$$\Rightarrow 27a^3x^3 + 27a^2x^2b + 9axb^2 + 27a^2d = 0$$
$$\Rightarrow (3ax)^3 + 3(3ax)^2b + 3(3ax)b^2 + b^3 - b^3 + 27a^2d = 0$$
$$\Rightarrow (3ax + b)^3 = b^3 - 27a^2d$$
$$\Rightarrow 3ax + b = \sqrt[3]{b^3 - 27a^2d} = p, \qquad (say)$$

Here p denotes any one value of $(3ax + b)$ then the three values of $(3ax + b)$ are $p, \omega p, \omega^2 p$ where $\omega(= -\frac{1}{2} + \frac{\sqrt{3}}{2}i)$ and $\omega^2(= -\frac{1}{2} - \frac{\sqrt{3}}{2}i)$ are imaginary cube root of unity.

Therefore, the roots of the given equations are

$$\frac{p-b}{3a}, \frac{(\omega p - b)}{3a}, \frac{(\omega^2 p - b)}{3a}$$

i.e., $\frac{p-b}{3a}, \left\{-\frac{1}{3a}\left(\frac{p}{2} + b\right) + i\frac{p}{2\sqrt{3}a}\right\}, \left\{-\frac{1}{3a}\left(\frac{p}{2} + b\right) - i\frac{p}{2\sqrt{3}a}\right\}$

The equation has one real root and a pair of complex conjugate root. Here the real root is given by the expression

$$\frac{\sqrt[3]{b^3 - 27a^2d} - b}{3a} \text{ i.e., } \frac{-b + \sqrt[3]{b^3 - 3^3a^{3-1}d}}{3a}.$$

Observation 1

The sign of the coefficient of x depends on the sign of the coefficient of x^3 and according to sign the equation can have the following variations

$$ax^3 + bx^2 + \frac{b^2}{3a}x + d = 0$$
$$ax^3 - bx^2 + \frac{b^2}{3a}x - d = 0$$
$$ax^3 + bx^2 + \frac{b^2}{3a}x - d = 0$$
$$ax^3 - bx^2 + \frac{b^2}{3a}x + d = 0$$

Observation 2

The general quadratic equation is given as
$ax^2 + bx + c = 0$,
where $a(\neq 0), b, c$ are real numbers (3.2)
Indian mathematician Sridharacharya had devised a formula for finding the roots of this equation (3.2) which is written as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\left(= \frac{-b \pm \sqrt{b^2 - 2^2a^{2-1}c}}{2a}\right)$$

This formula is known as Sridharacharya's formula or quadratic formula.
Now with close observation some similarities can be noted as following:-

- First term in both the expression is the coefficient of the term with second highest power with a negative sign i.e., the coefficient of x in case of quadratic equation and the coefficient of x^2 in case of cubic equation .
- In case of quadratic equation the next term in the numerator is under square root where as in case of cubic equation the next term is under cubic root.
- In case of quadratic equation the first term under square root is $(coefficent\ of\ x)^2$ where as for the cubic equation it is $(coefficient\ of\ x^2)^3$. Then for quadratic equation there is $(2^2a^{2-1}c)$ and for the cubic equation it is $(3^3a^{3-1}d)$. Again c and d are respectively the coefficient of x^0 in each case and a is the coefficient of x^2 in case of quadratic equation and coefficient of x^3 in case of cubic equation.
- In case of quadratic equation the denominator is $(2 \times coefficient\ of\ x^2)$ and for the cubic equation it is $(3 \times coefficient\ of\ x^3)$ i.e. denominator is na^n where a is the coefficient of x^n for $= 2,3$.
- Overall both the expressions are similar with few alternations. Number of terms in both the expressions are same.

The other two roots of the equation (3.1) can also be determined by using the relationship between the roots and coefficients of an equation. Let q, s be other two roots of the equation. Then

$$p + q + s = -\frac{b}{a} \qquad (3.3a)$$

$$pq + qs + sp = \frac{b^2}{3a^2} \qquad (3.3b)$$

$$pqs = -\frac{d}{a} \qquad (3.3c)$$

Here are two unknowns and three equations involving them, so any two equation can be chosen to determine the unknowns. Working with equation (3.3a) and (3.3c) seems more convenient to me so here this two equations are considered.

Equation (3.3c) gives, $q = -\frac{d}{aps}$

Using the values of q the equation (3.3a) becomes,

$$-\left(\frac{d}{aps} - s\right) = -\left(\frac{b}{a} + p\right)$$

$$\Rightarrow d - aps^2 = aps\left(\frac{b}{a} + p\right)$$

$$\Rightarrow d - aps^2 = bps + ap^2s$$

$$\Rightarrow aps^2 + (ap + b)ps - d = 0$$

Using the Sridharacharya's formula

$$s = \frac{-(ap + b)p \pm \sqrt{(ap + b)^2p^2 + 4apd}}{2ap}$$

Now if $s = \frac{-(ap+b)p+\sqrt{(ap+b)^2p^2+4apd}}{2ap}$ then $q = \frac{-(ap+b)p-\sqrt{(ap+b)^2p^2+4apd}}{2ap}$ and if $q = \frac{-(ap+b)p+\sqrt{(ap+b)^2p^2+4apd}}{2ap}$ then

$$s = \frac{-(ap + b)p - \sqrt{(ap + b)^2p^2 + 4apd}}{2ap}$$

Solving_the_Equation_Using_Cardan's_Method

Rewriting the equation (3.1)

$$ax^3 + bx^2 + \frac{b^2}{3a}x + d = 0 \qquad (3.1)$$

Let $x = y + h$,where h is such a quantity that coefficient of y^2 will be zero in the transformed equation. Then substituting the value of x the equation transformed to

$$a(y + h)^3 + b(y + h)^2 + \frac{b^2}{3a}(y + h) + d = 0$$

$$\Rightarrow a(y^3 + 3y^2h + 3yh^2 + h^3) + b(y^2 + 2yh + h^2) + \frac{b^2}{3a}(y + h) + d = 0$$

$$\Rightarrow ay^3 + (3ah + b)y^2 + \left(3ah^2 + 2bh + \frac{b^2}{3a}\right)y + \left(ah^3 + bh^2 + \frac{b^2}{3a}h + d\right) = 0 \qquad (3.4)$$

By the given condition,

$$3ah + b = 0 \Rightarrow h = -\frac{b}{3a} \qquad (3.5)$$

Using (3.5), equation (3.4) transforms to

$$ay^3 + \left\{3a \times \left(-\frac{b}{3a}\right)^2 + 2b \times \left(-\frac{b}{3a}\right) + \frac{b^2}{3a}\right\}y + \left\{a \times \left(-\frac{b}{3a}\right)^3 + b \times \left(-\frac{b}{3a}\right)^2 + \frac{b^2}{3a} \times \left(-\frac{b}{3a}\right) + d\right\} = 0$$

$$\Rightarrow ay^2 + \left(\frac{b^2}{3a} - \frac{2b^2}{3a} + \frac{b^2}{3a}\right)y - \frac{b}{27a^2} + d = 0$$

$$\Rightarrow ay^3 = \frac{b}{27a^2} - d$$

$$\Rightarrow y = \sqrt[3]{\frac{b}{27a^3} - \frac{d}{a}} = \sqrt[3]{\frac{b - 27a^2d}{27a^3}} = \frac{\sqrt[3]{b - 27a^2d}}{3a}$$

Let $p_1 = \frac{\sqrt[3]{b-27a^2d}}{3a}$ be one value of y. Then all the values of y are $p_1, \omega p_1, \omega^2 p_1$ where $\omega(= -\frac{1}{2} + \frac{\sqrt{3}}{2}i)$ and $\omega^2(= -\frac{1}{2} - \frac{\sqrt{3}}{2}i)$ are imaginary cube root of unity.

So the required roots of the equation (3.1) are $p_1 - \frac{b}{3a}, \omega p_1 - \frac{b}{3a}, \omega^2 p_1 - \frac{b}{3a}$

While solving any cubic equation $ax^3 + bx^2 + cx + d = 0$ with $a \neq 0$ using Cardan's method first the cubic equation is first converted in to depressed cubics of form $y^3 + 3Hy + G = 0$ by applying suitable substituting such that $x = y + h$,where h is such a quantity that coefficient of y^2 will be zero in the transformed equation. Then another substitution $y = u + v$ are made to transform the depressed cubic into a quadratic equation and hence after finding roots for the quadratic equation the roots of the original cubic equations are obtained after reversing the substitutions with basic calculations. But here after applying the suitable substitution the equation (3.1) transformed into the form $y^3 + G = 0$, which is the most simpliest form of cubic equation. Hence the process of further substitution and reversing the effects of substitutions are not needed here.

Nature_of_the_Roots

The roots of the equation (3.1) are

$$\frac{\sqrt[3]{b^3 - 27a^2d} - b}{3a}, \left\{-\frac{1}{3a}\left(\frac{\sqrt[3]{b^3 - 27a^2d}}{2} + b\right) \pm i\frac{\sqrt[3]{b^3 - 27a^2d}}{2\sqrt{3}a}\right\}$$

Here the possibilities are

- All the three roots are real.
- One root is real and a pair of complex conjugate root.

Considering the case (i) i.e., all three roots are real. For this case to be true, the must condition is

$$\frac{\sqrt[3]{b^3 - 27a^2d}}{2\sqrt{3}a} = 0$$

$\Rightarrow \sqrt[3]{b^3 - 27a^2d} = 0$, since here $a \neq 0$ is always true.

$$\Rightarrow b = 3\sqrt[3]{a^2d}$$

Substituting the above value of b in equation (3.1) it is obtained that

$$ax^3 + 3\sqrt[3]{a^2d}x^2 + 3\sqrt[3]{ad^2}x + d = 0$$

$$\Rightarrow (\sqrt[3]{a}x + \sqrt[3]{d})^3 = 0$$

$$\Rightarrow x = -\sqrt[3]{\frac{d}{a}}, -\sqrt[3]{\frac{d}{a}}, -\sqrt[3]{\frac{d}{a}}$$

So when there will be three real roots then the roots are all equal. Further the roots are positive if sign of a, d are different and roots are negative if sign of a, d are same.

Considering the case (ii) i.e., One root is real and a pair of complex conjugate roots. This case is generally true.

The real root is $\frac{\sqrt[3]{b^3-27a^2d}-b}{3a}$ and it is positive when $d < 0$ where as it is negative when > 0 . So the sign of real root depends on the coeffecient of x^0 and if $d = 0$ then the real root becomes zero.

The pair of complex conjugate root is $\left\{-\frac{1}{3a}\left(\frac{\sqrt[3]{b^3-27a^2d}}{2} + b\right) \pm i\frac{\sqrt[3]{b^3-27a^2d}}{2\sqrt{3}a}\right\}$,where the real part is $\left\{-\frac{1}{3a}\left(\frac{\sqrt[3]{b^3-27a^2d}}{2} + b\right)\right\}$ and the imaginary part is $\frac{\sqrt[3]{b^3-27a^2d}}{2\sqrt{3}a}$.

The real part is positive when $b < \sqrt[3]{3a^2d}$ and negative when $b > \sqrt[3]{3a^2d}$.

Examples

(1)Finding the roots for the equation $13x^3 + 7x^2 + \frac{49}{39}x - 53 = 0$

Solution: $13x^3 + 7x^2 + \frac{49}{39}x - 53 = 0$

$$\Rightarrow 13x^3 + 7x^2 + \frac{7^2}{3 \times 13}x - 53 = 0$$

$$\Rightarrow 13^2x^3 + 13 \times 7x^2 + \frac{7^2}{3}x - 53 \times 13 = 0$$

$$\Rightarrow 13^3x^3 + 3 \times (3x)^2 \times \frac{7}{3} + 3 \times 13x \times \left(\frac{7}{3}\right)^2 + \left(\frac{7}{3}\right)^3$$

$$= \left(\frac{7}{3}\right)^3 - 53 \times 13^2$$

$$\Rightarrow \left(13x + \frac{7}{3}\right)^3 = \left(\frac{7}{3}\right)^3 - 53 \times 13^2$$

$$\Rightarrow 13x + \frac{7}{3} = \frac{\sqrt[3]{7^3 + 27 \times 53 \times 13^2}}{3} \approx 20.777$$

Here one value of $\left(13x + \frac{7}{3}\right)$ is 20.777 and hence all the other three values are 20.777, 20.777ω ,

$20.777\omega^2$ where $\omega(=-\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ and $\omega^2(=-\frac{1}{2}-\frac{\sqrt{3}}{2}i)$ are imaginary cube root of unity.

Therefore the required roots of the given equation are 1.418 , $-10.388 \pm i17.993$

(2)Finding the roots for the equation $3x^3-5x^2+\frac{25}{9}x-23=0$

Solution: $3x^3-5x^2+\frac{25}{9}x-23=0$

$$\Rightarrow 3x^3-5x^2+\frac{5^2}{3\times 3}x-23=0$$

$$\Rightarrow 3^2x^3-3x^2\times 5+\frac{5^2}{3}x-23\times 3=0$$

$$\Rightarrow 27x^3-3\times(3x)^2\times\frac{5}{3}+3\times 3x\times\left(\frac{5}{3}\right)^2-\left(\frac{5}{3}\right)^3 = 23\times 3^2-\left(\frac{5}{3}\right)^3$$

$$\Rightarrow \left(27x-\frac{5}{3}\right)^3=\frac{27\times 3^2\times 23-125}{27}$$

$$\Rightarrow 27x-\frac{5}{3}=\frac{\sqrt[3]{27\times 3^2\times 23-125}}{3}\approx 5.871$$

Here one value of $\left(25x-\frac{5}{3}\right)$ is 5.871 and hence all the other three values are 5.871, 5.871ω , $5.871\omega^2$ where $\omega(=-\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ and $\omega^2(=-\frac{1}{2}-\frac{\sqrt{3}}{2}i)$ are imaginary cube root of unity.

Therefore the required roots of the given equation are 0.302, $-2.936 \pm i5.084$

IV. CONCLUSION

In this paper a particular type of cubic equation has been stated and studied throughly. It is seen here that the formula derived for the discussed equation has similarities with the formula obtained by Indian mathematician Sridharacharya for the quadratic equations. The derived formula is much simpler to memorize and recall when needed. And will be helpful for solving this equations effortlessly. More over the discussed type of equation is also very easy to solve by using Cardan's method as after the very first substitution the equation transformed into the most simplest form. The nature of the roots for the discussed type of equation has been discussed in this paper and also all possible variants of this type of equation has been mentioned here.

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