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# Integral Time Counter Simulations of GARCH for Option Pricing

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**Abstract:** This paper displays derivation of the univariate and multivariate GARCH (1,1) model. The Stochastic Discount Factor and conditional Esscher transform were used as techniques to derive the models. The matrix discretized form of the multivariate equation is also displayed in this paper with a proposed error correction in the normalized and variance matrices. Standard assumptions on the parameter were critically assigned and set to ensure convergences and stability for the models. The simulations in the risk-neutral processes shown approximate values as Mont Carlo simulations in Duan (2000) with values of the standard deviation ranging from [0.0999, 0.5623] and payoff European call options [0.00, 1.1692]. Using python and R programming tools, simulations showed that the risk neutral processes and the multivariate GARCH (1,1) can be used to predict returns and even price derivatives.

**Key words:** GARCH (1,1), martingales, stochastic processes, risk-neutral measures, volatility, multivariate, running and conditional correlation.

## 1.1. INTRODUCTION

The drift for pricing optional derivatives began in 20$^{th}$ century with the creation of the Black-Scholes model in 1973 that evolutionary changed the world of financial markets. A most valued theoretical framework used by traders and investors to purchase, sell or invest in portfolios respectively in the trading markets. The importance in pricing derivatives under the premise of continuity property of the volatility has been offered by Black and Scholes (1973). Nevertheless, empirical data, according to Bakshi et al. (1997), indicates that asset volatility is not constant but rather varies over time, a phenomenon known as stochastic volatility. Accurately pricing options in real-world markets requires an understanding of stochastic volatility and the incorporation of this volatility into option pricing models. The early works attempted to capture stochastic volatility focused on continuous-time models, such as the work of Cox and Ross (1976) and Cox et al. (1985), which laid the foundation for understanding the structure of interest rates and option pricing under various stochastic processes. However much the models would predict the approximate returns, there were more un answered questions, Heston (1993) proposed the discrete-time models have gained popularity due to their computational flexibility and tractability in capturing empirical features of asset price dynamics.

In the literature of Brennan (1979) introduced discrete-time models for pricing contingent claims, paving a way for further developments in option pricing theory. Building on to this framework, researchers have concatenated stochastic volatility into discrete-time models to better reflect the dynamics of financial markets. One notable approach is the Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) model proposed by Bollerslev (1986), which calls for time-varying volatility dynamics.

Heston and Nandi (2000) provide a framework which accounts for multiple time lags in the variation of the variance dynamics and permits the relationship between variance and the current asset's returns in addition. By including Heston (1993) closed-form stochastic volatility model as a continuous-time limit, the single lag version of the model brings together the discrete-time modeling of GARCH with the continuous-time random volatility approach to option valuation.

In light of these developments, this research aims to further study and explore the role of stochastic dynamics in discrete-time pricing derivatives, with a focus on incorporating Multivariate GARCH model. With all the literature, challenges persist in estimating model parameters and evaluating model performance accurately, as highlighted by Bakshi et al. (1997) and Zhang (2024). Consequently, there is an urgent need for tackling both practical and theoretical problems byimproving discrete-time option pricing models that accurately account for stochastic volatilitydynamics. This paper is aimed at extending the existing models by including GARCH drifts andrefining parameter estimation methods, provide more insights on the Multivariate GARCH (1,1) and one dimensional GARCH models ultimately increasing the accuracy of the pricing in dynamic financial environments. The determination of parameters as proposed by Bollerslev (1986) governing the risk-neutral dynamics of GARCH model holds paramount significance in financial modeling and risk management, decision making of portfolio allocations and spillover effects of one market onto the other. Financial markets often exhibit correlations between price dynamics and patterns in volatility which is really unresponsive in daily market. In asset allocation under portfolio, it is more important to account for interconnection to preciously manage risk and make good decisions. Therefore, the approximate refining of the parameters. Black and Scholes (1973) proposed that estimating and refining of these parameters accurately helps financial institutions to enhance option pricing accuracy, Duan (1995) suggested that it refines risk management purpose of studying the dynamics of variance swaps in the daily market as Badescu and Kulperger (2008) indicated is to provide inward knowledge of merging the existing volatility of an asset as an index, not considering options. The study of this parameter technique, helps investors make decisions on when to benefit during the comparisons of the greatness of strike degree and volatility. In addition, Laurent (2021) stated that MuGARCH models provide a present platform for modeling of double-dimension volatility dynamics among multiple assets simultaneously by capturing the connections of their price movements. In this paper, we contribute to the ongoing discussions in estimating of parameters in the discrete-time setting of GARCH (1,1) option pricing and risk management.

## 2. Literature Review

### 2.1 Historical Background Of the literature On Option Pricing models

Creating option pricing models that can incorporate stochastic volatility and manage both discreteand continuous time frames has been a persistent area of attention in recent years. As an example, Heston et al. (2024) presented a novel discrete-time option pricing model with stochastic volatility that is closed-form. These models provide useful tools for pricing and hedging derivatives in dynamic market conditions by attempting to find a compromise between mathematical tractabilityand empirical applicability.

### 2.2 Option Pricing Models

In this section, we shall look at the common option pricing models: Black Scholes, Heston and Nandi GARCH models all used for option pricing.

#### 2.2.1 Black Scholes model.

This model was developed by Black and Scholes (1973) and stated as equation (2.2.4) provides prior knowledge for pricing European-style options. The fundamental assumption in this model is that there is continuous trading and constant volatility in the daily market. Black and Scholes (1973) derived the Black-Scholes differential equation, governing the dynamics of the option price. In fair options pricing, they constructed the model basing n the following conditions;

- short-time interest rate is constant and known before time for maturity.
- one could invest, save or borrow money with short-time interest rate.
- The stock price, $s_t$ is based on Ito's process with the constant drift and volatility constants.

The above assumptions provide the equation of the stock price as

$$ds_t = \mu s_t dt + \delta s_t dw_t \qquad (2.2.1)$$

where $\mu$ and $\delta$ are the respect constant coefficients of the drift/deterministic and wiener process $w_t$ is the random wiener process. Applying Ito's equation [1] on the considered derivative option written on the asset with price. Equation (2.2.1) represents the dynamics of the option over time. The solution to the equation is given as

$$s_t = s_0\left[\left(\mu - \frac{1}{2}\delta^2\right)\right]t + \delta w_t \qquad (2.2.2)$$

The proof can be found in the appendix (7.1). Note that in the proof there is no corollaries and lemma. It consists direct derivation. Using theorem (2.51), the mathematical computations show that the Black Scholes option pricing model becomes

$$C(S,t) = S_t F(d_1) - Xe^{-r(T-t)}F(d_2) \qquad (2.2.4)$$

Where:

$$d_1 = \frac{\ln(S_t / X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

is the constant interest rate, $F$ (·) is the cumulative distribution function of standard normal, $S_t$ is the stoke price, X is the strike price and T is the expiry time or time to maturity, $\sigma$ in $d_1$ and $d_2$ is the volatility of the underlying asset. In the work, Cox and Ross (1976) extended equation (2.2.1) to derive the option pricing formula using the principles in risk-neutral measure and considering alternative random processes for asset prices. Their contributions primarily focused on extending the model to accommodate discrete- time settings which addressed important limitations of the original continuous-time Black-Scholes model. In addition, Brennan (1979) explored the pricing of contingent

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[11] Consider the distribution such that $X_{t+dt} = X_t + \mu_t dt + \sigma_t \sqrt{dt}\, z$ where $z$ is a standard Gaussian, then perform Taylor expansion to obtain the equation

$$f(X_{t+dt}) = \left( f(X_t) + f'(X_t)\mu_t dt + \frac{1}{2} f''(X_t)\sigma_t^2 dt + o(dt) \right) + \left( f'(X_t)\sigma_t \sqrt{dt}\, z + \frac{1}{2} f''(X_t)\sigma_t^2 (z^2 - 1)dt + o(dt) \right)$$

claims in discrete-time models. He derived the option pricing formula using a binomial tree approach:

$$C = \sum_{j=0}^{N} \binom{N}{j} p^{j}(1-p)^{N-j} \max\{0, S_0 u^{j} d^{N-j} - K\} \quad (2.2.5)$$

Equation (2.2.5) represents the price of the European call option in a discrete-time setting, where $N$ is the number of time steps, $p$ is the probability of an up movement, $u$ and $d$ are the up and down factors in the binomial tree model, and $S_0$ is the current price of the underlying asset. The reasons for this could be as suggested in [1].

Black and Scholes (1973) assumed a non-varying volatility while MoralesBan˜uelos et al. (2022) and Christie (1982) assumed volatility as a deterministic function of time. The assumptions made in these models are violated with strong evidences by many empirical studies that the volatility surface displays skew effects. It is not clear why volatility exhibits this effect, but rather many explanations show that the returns in a market have high positive kurtosis (greater than 3) than normal values.

### 2.2.2 GARCH(p,q) model.

In everyday markets, the volatility randomly shifts and therefore all models with defined terms of this volatility have to exhibit the reflection of this domestic randomness. The assumption of a constant volatility in Black-Scholes however doesn't account for the empirical observations which suggests that volatility exhibits clustering and time-varying behavior Christie (1982). To address this limitation, researchers have developed stochastic volatility models such as Heston (1993) model, which assumes volatility follows a mean-reverting process. Despite its flexibility, the Heston model lacked analytical tractability, motivating the search for alternative approaches. The GARCH *(p, q)* model was represented as:

$$\sigma_k^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{k-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{k-j}^2 \quad (2.2.6)$$

Where: $\sigma_k^2$ is the conditional variance at time t, $\boldsymbol{\alpha_0}$ is a constant, $\boldsymbol{\alpha_i}$ are the coefficients of the past squared error terms, $\boldsymbol{\beta_j}$ are the coefficients of the past conditional variances, $\boldsymbol{p}$ indicates how many past squared error terms are included in the conditional variance equation to capture the persistence of volatility, $\boldsymbol{q}$ indicates how many past conditional variances are included in the conditional variance equation to capture the short-term effects on volatility. There are many different formulations that came up after the discovery of the GARCH model. This paper is aimed to focus on GARCH frameworks for option pricing. In the market setting, GARCH option pricing models in Duan (1995) provided a better suit for an interesting alternative to stochastic volatility option pricing models with martingale probability set of distributions $\nu = \{Q, P\}$ and the conditional variance as

$$h_k^{\nu} = \omega + \sum_{i=1}^{p} \beta_i h_{k-i\delta} + \sum_{i=1}^{q} \alpha_i (\xi_{k-i\delta})^2, \quad \xi \sim \mathcal{N}(0,1) \quad (2.2.7)$$

Heston (1993) proposed that for any asset associated to a risky in a discrete time setting $\{k \mid k = 1, \cdots, t-1\}$ the continuously

[1] The Black-Scholes model assumes that the underlying asset's returns are normally distributed, which implies a log-normal distribution for the asset prices. When the market exhibits a volatility change, the models exhibit non-log-normal distributions of the returns. The disparity, including shifts in market sentiment, dynamics of supply and demand, or the effect of market events on option pricing.

compounded return process can be structured as [1]

$$y_k^{\nu} = \log \frac{S_{k+\delta}}{S_k} = r + \lambda h_k + \sqrt{h_k}\xi_k \qquad (2.2.8)$$

With $\lambda$ is the market price of risk, $\mathcal{Q}$ and $\mathcal{P}$ in $\nu$ and martingale measure, $\mathcal{Q}$ which depends on the historical probability, $\mathcal{P}$. $S_k$ is the pricing process. Using this model, the subsequent contingency claims [2] cannot currently be valued because it is unknown how risk neutral measures will be distributed. This shall be discussed in the coming chapters. The necessary and sufficient condition for the presence of a stationary solution to the GARCH model was developed by Bollerslev (1986).

**2.2.3 Theorem.** *The GARCH (p ,q) mode stated in equation (2.2.6) is steady if*

$$\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j < 1 \qquad (2.2.9)$$

Stability is necessary for the GARCH model to effectively model and analyze financial volatility by ensuring that the model's estimates are precise, dependable, and understandable, which increases the model's value for understanding and managing risk in financial markets.

**2.2.4 Affine Heston Nandi GARCH models.** According to the historical probability, Alexandru and Ortega considered Equation (2.2.8) to be the discretized version of the Heston and Nandi (2000) model. [3] For trade dates with equal sub interval, they took into consideration that Δ = 1 . Alexandru and Ortega literature indicates that the general discretized dynamics of an affine version of GARCH*(1,1)* may be found using;

$$y_k = f_1(h_k, \theta(\Delta)) + \sqrt{\Delta}\sqrt{h_{k\Delta}}\xi_k \qquad (2.2.10)$$

$$h_{k+1} = f_2(h_k, \theta(\Delta)) + f_3(h_k, \xi_k, \theta(\Delta)). \qquad (2.2.11)$$

In this case, $\xi_k$ is a series of $\mathcal{F}_{k-1}$ field condition to identically independent random variables with a finite moment generating equation and to guarantee the positiveness and stationary conditions of variance actions, a vector parameter $\theta$ adhere to certain conditions. Using equation (2.2.6), we can see that $f_1$ and $f_2$ are affine in $h_k$, and that the innovation distribution determines the shape of the news function $f_3$.

Alexandru and Ortega interpreted the conditional variance process in $\mathcal{F}_{k-1}$ predictable process to be the only component generating the log-return $y_k$ dynamics defined as

$$h_k^{\mathcal{P}} = \mathrm{Var}^{\mathcal{P}}\left[y_k \mid \mathcal{F}_{k-1}\right] \qquad (2.2.12)$$

### 2.2.5 Multivariate HN GARCH model.

In this section, we represent the multivariate model, and it's existing cumulant generating function. Escobar et al. (2019) and Alexandru and Ortega proposed an assumption that the model is affine

***2.2.6* Assumption.** *If the joint cumulant generating function exists and affine inform* $\mathcal{C}^{\mathcal{P}}_{(y_k, h_{(k+1)})|\mathcal{F}_{(k-1)}}(\rho,\pi) = \tau_{(k-1,k)}(\rho,\pi) + \chi_{(k-1,k)}(\rho,\pi)h_k$ *(2.2.13) then it is structured and defined as* $\mathcal{C}^{\mathcal{P}}_{(y_k, h_{(k+1)})|\mathcal{F}_{(k-1)}}(\rho,\pi) = \mathrm{E}^{\mathcal{P}} \log\left[\exp\left(\rho y_k + \pi h_k \mid \mathcal{F}_{k-1}\right)\right]$

[1] The equation (2.2.8) works for some risk premium parameter λ in a martingale historical probability, and martingale measure

[2] derivatives whose future payoffs depend on the values of another underlying assets or is dependent on the realization of some uncertain future event.

[3] Affine GARCH models often extend the traditional GARCH framework by incorporating additional parameters.

*(2.2.14) where* $\tau_{(k-1,k)}(\rho,\pi)$ *and* $\chi_{(k-1,k)}(\rho,\pi)$ *are real valued functions with respective martingale states*

The assumption for this definition is justified by following the next proposition, which states that this martingale measure has to have a moment generating function given a field of market information.

In this paper is aimed at deriving the parameters of risk neutral measures for the general discrete time of equation (2.2.10) following the exponential pricing kernel, we also derive the risk neutral dynamics under the stochastic discount factor and the conditional Esscher transform tool commonly used for actuarial mathematicians.

## 2.3 Exponential pricing kernel

Understanding, assessing, and managing several kinds of risks in financial markets, such as volatility risk, is made possible by setting the best pricing kernel. By considering investors' expectations and risk preferences, the pricing kernel guides investment decisions and risk management strategies in a turbulent and dynamic market environment. Escobar et al. (2019), Alexandru and Ortega literature suggested the approach for determining the exponential pricing kernel is projected from the papers Christoffersen et al. (2013) and Majewski et al. (2015) for determining the cumulant generating function of the affine GARCH model following lemma (3.0.1) The pricing equation [1] is a pricing kernel. The accompanying risk prices, $\overline{\wedge}_1$ and $\overline{\wedge}_2$, and their corresponding effects on the pricing kernel are reflected in the formula, which accounts for changes in variance and equity in the context of lemma (3.0.1) presented in the chapter (3).

## 2.4 Martingale measure

In a real market, we have to believe that asset prices approximate the behavior of a martingale at a minimum of a short term. The accuracy hypothetical statement in the market is that the update information is immediately transited to stock value, so expected value of the stock tomorrow absolutely should be the value today [2].

**2.4.1 Definition** (*Martingales*). *For the probability space S with a sequence of real random variables* $X_0, X_1, X_2, \ldots$ *Interpret* $X_i$ *as the price of the asset at the* $i$ *th time step, then the sequence* $X_n$ *is a martingale if* $\mathbb{E}[|X_n|] < \infty$ *and* $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ **(2.4.1)**
*for all n.*

In simple terms we may say a martingale of price of an asset is defined as "Given all I know today, expected price tomorrow is the price today". For a continuously compounded returns, Heston and Nandi (2000) mentioned that any portfolio's discounted price can be determined by substituting the derived assumption from equation (2.4.1) and equation (2.5.3) with risk rate, $r$ in the martingale $\mathcal{Q}$ i.e.

$$\mathrm{E}^{\mathcal{Q}}\left[y_k \mid \mathcal{F}_{k-1}\right] = \exp(r) \qquad \textbf{(2.4.2)}$$

For the deep characterization in martingale measure $\mathcal{P}$, if there exists a moment generating function say $M_{y_k}^{\mathcal{Q}} = E[e^{\vartheta_k h_k}] < \infty, k \geq 1$

[1] The pricing exponential equation is given by

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_T} := \exp\sum_{k=1}^{T\times n}\left[\overline{\wedge}_1(\Delta)y_{k\Delta} + \overline{\wedge}_2(\Delta)h_{k\Delta} - C(y_{k\Delta}, h_{k\Delta})\left(\overline{\wedge}_1(\Delta), \overline{\wedge}_2(\Delta) \,\Big|\, \mathcal{F}_{(k-1)\Delta}\right)\right]$$

[2] If in most cases asset prices are lower than expected, arbitrageurs statistically would step in to buy the asset, and if its higher the discrepancy between the observed price and the expected value is eliminated

for some constants $\vartheta_1, \vartheta_2, \cdots$ then the stochastic sequence $(Z_n)_{n\geq 1}$ is called a martingale with

$$Z_0 = 1 \qquad (2.4.4)$$

and

$$Z_n = \prod_{k\leq 1} \frac{\exp \vartheta_k y_k}{\mathbb{E}^{\mathcal{P}}\left[\exp \vartheta_k y_k \mid \mathcal{F}_{k-1}\right]}, n \geq 1. \qquad (2.4.5)$$

The significance of properties (2.4.4) and (2.4.5) becomes particularly evident when undergoing a measure change.

**2.4.2 Theorem.** *Suppose that the set of all equivalent martingale measures, $\mathcal{P}$ is nonempty. Then the family of arbitrage-free prices at time $k$ of a derivative security with payoff $h(S_k)$ is non-empty and is given by equation (2.4.2.1)[1]:*

***Proof.*** The proof of the theorem follows immediately by replacing the simple expectations from the proof of Theorem 1.30 (see Follmer and Schied (2004), page 18-19) with conditional expectations.

## 2.5 Risk Neutral Distributions.

In a realistic market, models like equation (2.2.4) can be used to price options under the historical probability $\mathcal{P}$. In the risk-neutral markets, equivalent option pricing models are required with complete martingale continuous state, $\mathcal{Q}$. The generation of these equivalent models require the Girsanov theorem stated below. Elliott and Madan (1998) proposed that in state prices can expressed as conditional expectation as

**2.5.1 Theorem** (Girsanov Theorem)[2]**.** *For a wiener process $W_k$ with a historical probability $\mathcal{P}$, then there exists an equivalent martingale $\mathcal{Q}$ is defined by the ration.*

The discrete version of theorem is given by the equation

$$\frac{d\mathcal{Q}}{d\mathcal{P}} = \exp\left\{-\sum_{i=1}^{n}\Psi(X_i) - 0.5\sum_{i=1}^{n}\left[\Psi(X_i)\right]^2\right\} \text{with}$$

the change of martingale state for the continuous or discrete return process $y_k$.

**2.5.2 Conditional Risk-Neutral Measure.** let it be a risk neutral distribution subject to a martingale measure $\mathcal{Q} \in \nu$ the dynamics are given by simplified system of equation (2.2.8) and (2.2.7), such that

$$y_k^{\mathcal{Q}} = m_k + \sqrt{h_k}\xi_k \qquad (2.5.3)$$

where $m_k$ is the unknown parameter in the risk state

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[1] The equation of the security derivatives is given by

$$\Pi_t(h(S_T)) = \{\sup_{\mathcal{Q}\in\nu} \mathbb{E}^{\mathcal{Q}}\left[e^{-\sum_{k=k+1}^{T} r_k} h(S_T) \,\middle|\, \mathcal{F}_k\right]\}, \quad \text{where} \quad \mathcal{E}^{\mathcal{Q}}[h(S_T) \mid \mathcal{F}_k] < \infty$$

[2] The integral/ continuous version of the theorem is given by $\frac{d\mathcal{Q}}{d\mathcal{P}} = \exp\{-\Psi(X_k) - 0.5\left[\Psi(X_k)\right]\}$

Such that $X_k$ subsequent process and

$$W_k^{\mathcal{Q}} \& = W_k + \left[\Psi(X_k)\right], \quad \left[\Psi(X_k)\right] = \int_0^k X_s ds, \quad \Psi(X_k) = \int_0^k X_s dW_s \text{ subject to the assumption}$$

$$\mathrm{E}^{\mathcal{P}}\left[\int_0^T X_s^2 Z_s^2 ds\right] < \infty$$

$h_k^{\mathcal{Q}} = \omega + \beta_1 h_{k-\delta} + \alpha_1 \xi_{k-\delta}^2, \quad \xi \sim \mathcal{N}(0,1)$ (2.5.4)

The task here is to verify that $\mathcal{Q} \in \nu$ is a martingale measure which shall be discussed in next chapter. This paper focuses on discrete time frames of $y_k$ with $\delta = 1$

**2.5.3 Stochastic Discount Factor.** Hansen and Richard introduced the notion of the stochastic discount factor (SDF) in 1987. The stochastic process $M_k$ which is measurable with regard to the filtration $\mathcal{F}_k$ is known as an SDF. Alexandru and Ortega stated a version of the fundamental theorem of asset pricing, assuming that there is no arbitrage in the market. They demonstrated that the price of a contingent claim with a payout of $h(B_T)$ at time $k$. The associated with the family of SDF $(M_k)_{0 \le k \le T}$ is given by:

$$\Pi_{M_k}^{\mathcal{P}}(h(B_T)) = \mathbb{E}^P\left[M_{k+1} \ldots M_T h(B_T) \mid \mathcal{F}_k\right] \quad (2.5.5)$$

The no-arbitrage opportunities are eliminated by making the following assumptions.

$$\mathbb{E}^{\mathcal{P}}[M_k \mid \mathcal{F}_{k-1}] = e^{-r} \quad (2.5.6)$$

$$\mathbb{E}^{\mathcal{P}}[M_k e^{y_k} \mid \mathcal{F}_{k-1}] = 1 \quad (2.5.7)$$

In relation to the stochastic Discount Factor, Duan (1995) presented the Locally Risk Neutral Valuation Relationship (LRNVR) for pricing under the normality assumption for the stock innovations and postulated that the martingale measure $\mathcal{Q}$ with the LRNVR satisfies the following two conditions: $\mathcal{F}_{k-1}, Y_k$ and $\mathrm{Var}^{\mathcal{Q}}[Y_k \mid \mathcal{F}_{k-1}] = \mathrm{Var}^{\mathcal{P}}[Y_k \mid \mathcal{F}_{k-1}]$

The next theorem introduces the definition of a one-period stochastic discount factor (SDF) [1],drawing on the work of Hansen and Renault (2009). The following parameter limitation is obtained from definition in for pricing the return process $y_k$ in equation (2.2.8).

**2.5.3 Theorem** *One-period Stochastic Discount Factor [8] (SDF)).* *If $(M_k)_T$ for $k \in \{1, \ldots, T\}$ is a positive process, then $\sigma$-field of filtration $(\mathcal{F}_t)_k$ . One-period stochastic discount factor process hold if for $k \in \{1, \ldots, T\}$:*

$$\mathrm{E}^P\left[\frac{S_k}{S_{k-1}} M_k \mid F_{k-1}\right] = 1 \quad (2.5.8)$$

## 2.6 Conditional Esscher Transform.

The clear approaches to Esscher Transforms were introduced by Yang and Sui (2004), and was first introduced by Esscher (1932). Yang and Sui (2004) offered an elegant way to select an equivalent martingale measure in an incomplete market setting which takes a probability density $f(x)$ and transforms it to a new probability density $f(x;h)$ with a parameter $h$.

$$f(x;h) = \frac{e^{-ht} M_X(t)}{\int_{-\infty}^{\infty} e^{-ht} M_X(t) dt}, \quad (2.6.1)$$

where $M_X(t)$ $\quad M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

Here, $M_X(t)$ is the MGF of the original distribution.

## 3 Methods of Modelling

In this section, we show how to derive the risk neutral process, $y_k$ using various approaches. In addition, we also develop the multivariate cumulant generating function of the risk neutral measure. Alexandru and Ortega stated that in order to derive the risk neutral

[1] For a discount factor $\beta$, marginal utility of consumption at time, k and level of consumption, C then SDF is defined as: $M_k = \beta \frac{u'(S_{k+1})}{u'(S_k)}$

measures, we need the radon- Nykodime derivative for change of measure. This is very important thought the derivations.

**3.01 Lemma *Radon-Nykodime derivative.*** *Elliott and Madan (1998) and Follmer and Schied (2011) expressed $\mathcal{Q}$ to be a probability measure equivalent w.r.t. $\mathcal{P}$ on a $\sigma-$ field with a filter constructed by $\mathcal{G}_T$. For any $0 \le k \le T$, $Z_T$ is Radon-Nikodym derivative with properties*

- **P.1** The conditional Radon-Nikodym derivative of $\mathcal{Q}$ w.r.t. $\mathcal{P}$ on $\mathcal{G}_k$ is given by

$$Z_k := \frac{dQ}{dP}\Big|_{\mathcal{G}_k} = E^{\mathcal{P}}\left[\frac{dQ}{dP}\Big|_{\mathcal{G}_k}\right] \qquad (3.0.1)$$

In particular, $Z_k$ is a $\mathcal{G}_k$ mean one martingale under $\mathcal{P}$.

- **P2** For any $\mathcal{G}_s$ $(s \ge k)$ and $\mathcal{Q}$-integrable measurable function $g$, we have:

$$E^{\mathcal{Q}}[g \mid \mathcal{G}_k] = E^{\mathcal{P}}\left[\frac{Z_s g}{Z_k}\Big|_{\mathcal{G}_k}\right] \qquad (3.0.2)$$

The most important property is property (3.02) referred to in the conditional expectation for the Bayes rule in probability, whereas property (3.01) is a simple application of the tower property for the conditional expectations.

### 3.1 Risk neutral dynamics under SDF

The proceeding theorem demonstrates how to create a risk-neutral measure through observation made by follmer2011stochastic to describe the pricing of a contingent claim using the relations (2.5.5) and equation (2.4.6) in theorem (2.4.2).

**3.1.1 Theorem** *Let $(v_k)_{0\le k\le T}$ be a family of stochastic discount factors (SDF) satisfying the conditions specified in (2.5.3). Let $\mathcal{Q}$ be a measure defined by its density:*

$$\frac{dQ}{dP} := Z_T = e^{\sum_{k=1}^{T} r_k Y_k} \prod_{k=1}^{T} M_k, k \ge 1 \qquad (3.1.1)$$

*with (2.4.4). Then $\mathcal{Q} \in \nu$ and the prices in equation (2.5.5) and equation (2.4.6) are consistent.*

The objective is to demonstrate, by means of qualities (2.4.4) and (2.4.6), whether the corresponding probability measure $\mathcal{Q}$ is a martingale. Since the market is imperfect, as we previously said, there are a number of possible approaches to derive the risk-neutral dynamics using the stochastic discount factor. The utilization of equation under theorem (3.1.1) is strictly emphasized in this study.

### 3.2 Risk neutral dynamics under Duan's LRNV

In this section, we represent the format and main points that arise from this analysis. We start by deriving the equation of the returns with the risk neutral measures. Using equation (2.5.3) under the martingale measure $\mathcal{Q}$ and equation (2.2.8) under the historical probability $\mathcal{P}$. This approach of derivation is appropriate if the representative is an expected utility maximizer and utility function is time separable and additive, then LRNV (Locally Risk Neutral Valuation) holds under any of the following conditions;

- Utility function[1] is of constant risk aversion [2] and the changes in the aggregate consumption are distributed normally under $\mathcal{P}$
- The mean and variances are constant.[3]
- The utility function is linear.

### 3.2.1 Unknown parameter, $m_k$

We begin by considering the martingale measure $\mathcal{Q}$ and equation (2.4.2). The expectation of the log-returns shows that the unknown parameter is given $m_k$ shown below

$m_k = r - 0.5h_k$

*Proof. From equation (2.4.2) and using the log-normal definition, we have*

$$E^{\mathcal{Q}}\left[e^{y_k} \mid \mathcal{F}_{k-1}\right] = E^{\mathcal{Q}}\left[e^{m_k+\sqrt{h_k}z_k} \mid \mathcal{F}_{k-1}\right]$$

$$= e^{m_k+0.5\sqrt{h_k}}$$

Using equation (2.4.1), we have $e^r = e^{m_k+0.5\sqrt{h_k}}$ which implies that $r = m_k + 0.5\sqrt{h_k}$ thus $m_k = r - 0.5\sqrt{h_k}$

**3.2.2 Conditional variance.** The equation (2.5.4) is the general formula. Using the presumptions of \cite{Hansen} and \cite{heston2000}, the *i.i.d* random variable formula $\xi_k$ in $\mathcal{P}$ is given

$$\xi_k = (\lambda - 0.5)\sqrt{h_k} + z_k$$

*Proof* . The proof takes an assumption that for equilibrium, two dynamics in $\mathcal{P}$ and $\mathcal{Q}$ are such that

$$y_k^{\mathcal{P}} = y_k^{\mathcal{Q}}$$

$$r - \lambda h_k + \sqrt{h_k}\xi_k = m_k + \sqrt{h_k}z_k$$

$$r - \lambda h_k + \sqrt{h_k}\xi_k = r - 0.5h_k + \sqrt{h_k}z_k$$

$$\sqrt{h_k}\xi_k = (\lambda - 0.5)h_k + \sqrt{h_k}z_k$$

$$\xi_k = (\lambda - 0.5)\sqrt{h_k} + z_k$$

Substituting in the equation (2.5.4), we have,

$$h_k^{\mathcal{Q}} = \omega + \beta_1 h_{k-1} + \alpha_1\left[(\lambda - 0.5)\sqrt{h_{k-1}} + z_{k-1}\right]^2 \quad (3.2.1)$$

On substituting the unknown parameter $m_k$ is the conditional average return, the risk neutral dynamics gets exactly the same format as that of Heston and Nandi (2000) The Duan's LRNV can be discussed using the stochastic Discount approach.

### 3.3 Risk Neutral dynamics under Conditional Esscher Transform

Another version of the risk neutral measure is the conditional Esscher transform version. According to Buhlmann et al. (1996), who first suggested it for the generalized time jumps. To the little of my understanding, Gerber and Shiu (1994) were the first to use Esscher transforms in the context of option pricing for an incomplete market. In this section we use

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[1] let us assume that the dynamics of the risky asset in the Lucas economy with the innovations $z_t$ are Gaussian, and that the underlying utility function is isoelastic given

$$u(x) = \frac{x^{1-\gamma} - 1}{1-\gamma},$$

where $\gamma$ is e relative risk aversion coefficient.

[2] Risk aversion is the tendency to avoid risk. The term risk-averse describes the investor who chooses the preservation of capital over the potential for a higher-than-average return

[3] Constant mean and variance ensure the interest rate is also constant

the comparable articulation of the research of Gerber and Shiu (1994) which proposed the use of exponential affine parameterized form of the SDF. We assume that the moment generating function [1], $M_k$ of log return processes with respect to $\mathcal{F}_{k-1}$ is unique and exits such that equation (2.4.3) i.e.

$$M^{\mathcal{P}}_{y_k}(\vartheta) = E[e^{\vartheta_k y_k} \mid \mathcal{F}_{k-1}] < \infty, \quad k \geq 1, \quad \vartheta \in \mathcal{R} \quad (3.3.1)$$

If we defined stochastic process $Z_t$ such that properties (2.4.4) and (2.4.5) are verified. We defined a martingale $\mathcal{Q}$ such that its conditional Esscher transform with respect to $\mathcal{P}$ satisfies Radon-Nikodym condition (3.0.1). Since the Esscher transform is versatile for any probability distribution with unique existences of moment generating function, Yang and Sui (2004) defines the moment generating function of $\mathcal{Q}$ with respect to $\mathcal{P}$ as;

$$M^{\mathcal{Q}}_{k|F_{k-1}}(u,\vartheta) = \frac{M^{\mathcal{P}}_k \mid F_{k-1}(u+\vartheta_k)}{M^{\mathcal{P}}_k \mid F_{k-1}(\vartheta_k)} \qquad (3.3.2)$$

Within the literature on option pricing if we let parameter $u=1$ and $k$ be the time-counter parameter, , Yang and Sui (2004) and Badescu and Kulperger (2008) stated that the resulting equation

$$M^{\mathcal{P}}_k \mid F_{k-1}(1+\vartheta^*_k) = \exp r M^{\mathcal{P}}_k \mid F_{k-1}(\vartheta^*_k) \quad (3.3.3)$$

is subjected to a unique solution $\vartheta^*_k$ given by

$$\vartheta^*_t = \frac{1}{h_k}\left[r - m_k - 0.5h_k\right] \qquad (3.3.4)$$

This equation guarantees that the risk-free rate and the average log returns of the stock under martingale $\mathcal{Q}$ are identical and consistent. In this martingale, We defined from the beginning that the log-return process is given by equations (2.5.3) with the conditional variance defined in equation (2.5.4). Using equation (3.3.2), the algebraic computations show that the moment generating function of the returns is a log-normal distributions given by

$$M^{\mathcal{P}}_{k|F_{k-1}}(u,\vartheta) = \exp(m_k + h_k^2\vartheta^*_k)u + 0.5h_k^2u^2 \qquad (3.3.5)$$

On comparing with the definition of moment generating function, the mean equation of the returns were given by

$$E^{\mathcal{Q}}\left[y_k \mid \mathcal{F}_{k-1}\right] = m_k + h_k\vartheta^*_k$$
(3.3.6)

and conditional variance equation

$$\mathrm{Var}^{\mathcal{Q}}\left[y_k \mid \mathcal{F}_{k-1}\right] = h_k \qquad (3.3.7)$$

substituting equation (3.3.4) in both equations (3.3.6) and (3.3.7), the resulting expectation was found to be exactly the same with those of Heston and Nandi (2000). i.e.

$E^{\mathcal{Q}}\left[y_k \mid \mathcal{F}_{k-1}\right] = r - 0.5h_k$ and the conditional variance in equation (3.3.7)

We have shown that the Conditional Esscher transform gives equivalent results of the log-return formulas as Duan's LRNVR. The task to choose which one is appropriate was finished in Badescu and Kulperger (2008) paper which

[1] The moment-generating function (MGF) of a log-normal distribution is given by:

$$M_X(t) = E[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

here $\mu$ (the mean of the logarithm of $X$ and $\sigma$ (the standard deviation of the logarithm of $X$, $t$ is the parameter of the MGF.

gave an strong rationale for choosing Esscher transform over Duan's LRNVR, based on the fact that it is justified by a log-linear pricing kernel serves as a function on $y_k$.

### 3.4 Affine Multivariate GARCH model under historical measure $\mathcal{P}$

The outstanding constraint to GARCH models is assumption for single dimension time series which is not the case with common markets. In order to obtain the fine structure of the multi dimension affine model, we need to build specific parameters for the conditional variance provided we have the modification distribution. Alexandru and Ortega formulated limitation conditional covariances' joint cumulant generating function (cgf) in the invariant case. The distinct form of the Heston and Nandi (2000) risk neutral return process $y_k$ under the martingale $\mathcal{P}$ is assumed to be given by

$$y_k^{\mathcal{P}} = (r+\lambda h_k)\Delta + \sqrt{\Delta}\sqrt{\Delta h_k}\,\xi_k^P, \xi_k^P \sim \mathcal{N}(0,1) \quad (3.4.1)$$

The corresponding conditional variance (3.4.2) is given by

$$h_k^{\mathcal{P}} = \omega(\Delta) + \beta(\Delta)h_{k-1} + \alpha(\Delta)\left(\xi_{k-1}^P - \gamma(\Delta)\sqrt{h_{k-1}}\right)^2.$$

As assumed earlier in equation (2.2.10), we shall deal with these equations when time step $\Delta = 1$. Using equation (2.2.14)), we substituted the above equations and the expressed expected value on a historical probability in joint cumulant function of $\xi_k$ and $\xi_k^2$ is given in equation (3.4.3) [1]. Assuming that the parameters controlling the conditional variance dynamic in equation (3.4.2), meet the standard criteria, it will be probably certain that the model is stable. To illustrate, Alexandru and Ortega proposed that $\omega$, $\alpha$, and $\beta$ need to be non-negative. Additionally, the persistence stability must satisfy $\beta + \alpha\gamma^2 < 1$. The leverage impact is quantified by the measure $\gamma$ as mention in Escobar et al. (2019), a positive value of which suggests a negative connection, as is often seen in equities markets, between the volatility degrees and asset performances. Using the factor that the standard normal in $t$ defined by

$$e^{\rho t^2 + \pi t} = -\log\sqrt{1-2\pi} + \frac{\rho^2}{2(1-2\pi)} \quad (3.4.4)$$

Then algebraic computations show that the equations of the log-return under historical probability satisfy equation (2.2.13) with the coefficient

$$\tau_{(k-1)}^{\mathcal{P}}(\rho,\pi) = \rho r + \omega\pi - log\sqrt{1-2\alpha\gamma\pi} \quad (3.4.5)$$

and the conditional variance coefficient given by

$$\chi_{(k-1)}^{\mathcal{P}}(\rho,\pi) = \frac{(\rho - 2\alpha\gamma\pi)^2}{2(1-2\alpha\pi)} + \lambda\rho + \beta\pi + \alpha\gamma^2\pi$$

(3.4.6)

### 3.5 Affine Multivariate GARCH model under measure $\mathcal{Q}$

In this section, we object to understanding the normality parameters of the risk neutral measures. The risk neutral measurements are presented in this part, together with the condition relation equation for market prices of equity risks and variance in the payoff space.

Using assumption (2.2.13), we state that the proposition with respect to a martingale measure $\mathcal{Q}$, the risk neutral joint cumulant generating function is given by equation (3.5.1)

[1] The expectational equation is given by

$$E^{\mathcal{P}}\log\left[\exp\left(\rho y_k + \pi h_{(k+1)} \mid \mathcal{F}_{(k-1)}\right)\right] = \rho r + \omega\pi + \lambda\rho + \beta\pi + \alpha\pi\gamma^2 + E^P\left[\exp\alpha\pi\xi_k^2 + \left(\rho - 2\gamma\alpha\pi\right)\sqrt{h_k}\,\xi_k\right]$$

$$\mathcal{C}^{\mathcal{P}}_{(y_k, h_{(k+1)})|\mathcal{F}_{(k-1)}}(\rho,\pi) = \tau^{\mathcal{Q}}_{(k-1,k)}(\rho,\pi) + \chi^{\mathcal{Q}}_{(k-1,k)}(\rho,\pi) h_k$$

here the coefficients are given by equation (3.5.2) and equation (3.5.3) respectively

$$\tau^{\mathcal{Q}}_{(k-1,k)}(\rho,\pi) = \tau^{\mathcal{P}}_{(k-1,k)}(\rho+\overline{\wedge}_1, \pi+\overline{\wedge}_2) - \tau^{\mathcal{P}}_{(k-1,k)}(\overline{\wedge}_1, \overline{\wedge}_2)$$

$$\chi^{\mathcal{Q}}_{(k-1,k)}(\rho,\pi) = \chi^{\mathcal{P}}_{(k-1,k)}(\rho+\overline{\wedge}_1, \pi+\overline{\wedge}_2) - \chi^{\mathcal{P}}_{(k-1,k)}(\overline{\wedge}_1, \overline{\wedge}_2)$$

The evidence uses algebraic computations to demonstrate exactly the coefficients rely on the market price of risks subject to no-arbitrage restrictions,

$$\tau^{\mathcal{P}}_{(k-1,k)}(1+\overline{\wedge}_1, \overline{\wedge}_2) = \tau^{\mathcal{P}}_{(k-1,k)}(\overline{\wedge}_1, \overline{\wedge}_2) + r \quad (3.5.4)$$

$$\chi^{\mathcal{P}}_{(k-1,k)}(1+\overline{\wedge}_1, \overline{\wedge}_2) = \chi^{\mathcal{P}}_{(k-1,k)}(\overline{\wedge}_1, \overline{\wedge}_2) \quad (3.5.5)$$

**3.5.2 Proposition.** *Similarly, if we may just set initial values in the payoff market space*

$$\pi = 0,\ \rho = 1,\ \tau_{(k-1,k)}(\rho,\pi) = r,\ \chi_{(k-1,k)}(\rho,\pi) = 0$$

for equations (3.5.2) and (3.5.3), we obtain the no- arbitrage restrictions. For any market or payoff space for equation (2.3.1), two market risk-price combinations subject to equation (3.5.5) satisfy the following equation

$$\overline{\wedge}_1 = -\lambda - 0.5 + 2\alpha\overline{\wedge}_2(\lambda+\gamma) \quad (3.5.6)$$

**3.5.2 Conjecture** Suppose the asset return follow the affine GARCH dynamics under $\mathcal{P}$ equation (3.4.1). The risk neutral dynamics under variance dependent on the pricing kernel in equation (2.3.1) subject to equation (3.5.1) and proposition (3.5.2), the log-return in the risk-neutral measure is normally distributed i.e.

$$y_k^{\mathcal{Q}} \sim \mathcal{N}\left(r - 0.5h_k^*, h_k^*\right)$$

thus, the return risk neutral dynamics is given by

$$y_k^{\mathcal{Q}} = r - 0.5h_k^* + \sqrt{h_K^*}\,\xi_k^*, \quad \xi_k^* \sim \mathcal{N}(0,1) \quad (3.5.7)$$

$$h_k^* = \omega^* + \beta^* h_{k-1}^* + \alpha^*\left(\epsilon_{k-1}^* - \theta^*\sqrt{h_{k-1}}\right)^2 \quad (3.5.8)$$

With

$$\theta^* = \lambda(1-2\alpha\overline{\wedge}_2) + \gamma(1-2\alpha\overline{\wedge}_2) + 0.5$$

The risk neutral dynamics are then given by

$$\omega^* = \frac{\omega}{1-2\alpha\overline{\wedge}_2},\ \alpha^* = \frac{\alpha}{(1-2\alpha\overline{\wedge}_2)^2},$$
$$\lambda^* = \alpha(1-2\alpha\overline{\wedge}_2),\quad \gamma^* = \gamma(1-2\alpha\overline{\wedge}_2)$$

According to Alexandru and Ortega, one can utilize the conclusions in proposition (3.5.1) to describe the multi-step risk-neutral cumulant generating function of $Y_k$ and $h_k^*$ provided the GARCH dynamics under $\mathcal{P}$ and $\mathcal{Q}$ retain identical structure

$$\mathcal{C}^{\mathcal{Q}}_{(y_k, h^*_{(k+1)})|\mathcal{F}_{(k-1)}}(\rho,\pi) = \tau^*_{l,k}(\rho,\pi) + \rho Y_l + \chi^*_{l,k}(\rho,\pi) h^*_{l+1} \quad (3.5.9)$$

Here the coefficients $\tau^*_{l,k}(\rho,\pi)$ and $\chi^*_{l,k}(\rho,\pi)$ satisfy the following recursions for all $l < k$:

See footnote[1] for any arbitrary values $\rho$ and $\pi$, with given endpoint restrictions $\tau^*_{l,k}(\rho,\pi) = 0$ and $\chi^*_{l,k}(\rho,\pi) = \pi$. We can use this to obtain the distribution of shocks under the equivalent martingale measure. We now apply the change of measure to MuGARCH model with $\chi^*$ and $\tau^*$ given in equations in equations **3.5**

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[1] The coefficients are given by $\tau^*_{l,k}(\rho,\pi) = \tau^*_{l,k}(\rho,\pi) + \rho r + \omega^*\chi^*_{l,k}(\rho,\pi) \quad -\frac{1}{2}\log\left(1 - 2\overline{\wedge}^*\chi^*_{l,k}(\rho,\pi)\right)$

$$\chi^*_{l,k}(\rho,\pi) = -\frac{\rho}{2} + \left(\alpha^*(\gamma^*)^2 + \beta\right)\chi^*_{l,k}(\rho,\pi) + \frac{\left(\rho - 2\alpha^*\gamma^*\chi^*_{l,k}(\rho,\pi)\right)^2}{2\left(1 - 2\alpha^*\chi^*_{l,k}(\rho,\pi)\right)}$$

### 3.6 Vector parameters of Multivariate GARCH model

As there are many forms of multivariate GARCH models, this paper aims to focus on the Dynamic Conditional Correlation of Multivariate GARCH. Perhaps, we need to first construct the Constant Conditional Correlation multivariate model, consider the normalized residuals, $\xi \sim (0, \mathcal{I}_{n\times n})$ where $\mathcal{I}$ is an identity matrix and $n$ number of observations made, retrieved from the $n$ dimension models. [1] If $n=1$ we retrieve equation (2.2.10), then for bivariate model $n=2$ thus $\xi = \begin{pmatrix} \xi_{1,k} \\ \xi_{2,k} \end{pmatrix}$ and the covariance matrix can be given by $\begin{pmatrix} h_{1,k} \\ h_{2,k} \end{pmatrix}$

We can separate the conditional variances into the conditional standard deviations and constant using the technique proposed by Engle (2002) and Bollerslev (1990). While modeling the vector form, the conditional variance is considered as product of time invariant probabilities $\Lambda_i$ and the square difference between *i, j* observations of $k$ as time series. By natural definition of correlation, $\varphi_{ij}$ in terms conditional covariance, $h_k$ and variance is given $\varphi_{ij,k} = \frac{h_{ii,k}}{\sqrt{\sigma_{ii,k}\sigma_{jj,k}}}$ here $-1 < \varphi_{ij,k} < 1$ and $k$ is the already iteration of time series. In the literature of Silvennoinen and Terasvirta (2008) and Bollerslev (1990) assumed constant correlations between the and the conditional covariance matrix shall be given by $h_{ij,k} = \left(\sqrt{\sigma_{ii,k}}\sqrt{\sigma_{jj,k}}\right)\varphi_{ij}$ if we define the variance matrix conditioned to field space [2] as $\sigma_{ii,k} = \zeta_{i,k}^2 \Lambda_i$ where $\Lambda$ is time independent probability function and $\zeta$ is difference from the mean returns and the actual return from all observations, $i$. We can now retrieve the conditional covariance matrix with constant correlations as

$$h_{ii,k} = \sqrt{\mathbf{diag}\left(\zeta_{i,k}^2\right)}\sqrt{\Lambda_i}\varphi\sqrt{\Lambda_i}\sqrt{\mathbf{diag}\left(\zeta_{i,k}^2\right)}$$

where the $\sigma_{ii,k}$ is positive definite matrix of $diag(\zeta_{n,k}^2, \zeta_{2,k}^2, \cdots, \zeta_{n,k}^2)$, $\Lambda_i$ should also a positive definite matrix which propositions are very easy to take and verify in order to maintain the positive entries of the covariance matrix. The dynamic conditional correlation secures the assumption of time invariant is violated by allowing the correlation to vary on time for every observation $i$

$$h_{ii,k} = \sqrt{\mathbf{diag}\left(\zeta_{i,k}^2\right)}\sqrt{\Lambda_i}\varphi_k\sqrt{\Lambda_i}\sqrt{\mathbf{diag}\left(\zeta_{i,k}^2\right)} \quad (3.6.1)$$

The time dependent correlation matrix is positive definite given by

$$\partial_k = \left(\mathbf{diag}(\zeta_{i,k}^2)\right)^{-\frac{1}{2}}\nabla_k\left(\mathbf{diag}(\zeta_{i,k}^2)\right)^{-\frac{1}{2}}$$

(3.6.2)

where $\partial_k = \sqrt{\Lambda_i}\varphi_k\sqrt{\Lambda_i}$ and $\nabla_k$ are the positive definite correlation and covariance matrix respectively. Observe that this does not take into any consideration of the martingale measure $\mathcal{P}$ or $\mathcal{Q}$ but rather it is a generalized state.

### 3.7 Vector Parameters in Probability $\mathcal{P}$

We begin with a modification of returns behavior to determine our HN-MuGARCH algorithm with discrete time-varying

[1] Most of the work in this section is got from Engle (2002),Silvennoinen and Terasvirta (2008),Bollerslev (1990),Bollerslev (1986)

[2] We may define the $\phi_{k-1}$ as the field space of information generated up to time $k=t$

restriction. Specifically, Equation (2.2.8) can be represented in a matrix from as

$$y_{i,k}^{\mathcal{P}} = \mathrm{diag}(\{r_{ii}\}) + \mathrm{diag}(\lambda_{ii,k})h_{ii,k} + \sqrt{h_{ii,k}}Z_{ii,k}, \quad (3.7.1)$$

Where $i = 1, 2, ...., n$ and $\lambda_{ii,k} = \mathcal{I}_{i\wedge i} \wedge \lambda$

$r_{ii,k} = \mathcal{I}_{i\wedge i} \wedge r$. The return equation is therefore the matrix vector invariant affine GARCH models with conditional variances $h_{i,k}$. For each time-point requires an update to $h_k$, after which the coefficient matrix must be rebuilt using equation (3.6.2) and $\mathbf{diag}(\zeta_{i,k}^2)$, which contains the diagonal elements of $\nabla_k$. With this approach Bollerslev (1990) and Engle (2002) proposed that the covariance matrix, $h_k$ can then be rebuilt using equation (2.2.7) In order to guarantee that $h_k$ is positive definite, Escobar et al. (2019) and Silvennoinen and Terasvirta (2008) proposed further the format of $h_k$ using a GARCH*(1,1)*-like structure and enforcement stationary condition $\Gamma_1 + \Gamma_2 < 1$.

The conditional variance process in the historical probability setting is thus given by equation (3.7.2)

$$h_k = \overline{h_{1,k}} \wedge (1 - \Gamma_1 - \Gamma_2) + \Gamma_1 \wedge (Z_{1,k-1}Z_{1,k-1}^{\top}) + \Gamma_2 \wedge h_{1,k-1}, \quad Z_k \sim \mathcal{N}(0, \partial_k)$$

which may preciously be simplified by substituting scalars for the matrix structures to obtain fewer open parameters as equation (3.7.3)

$$h_k = \overline{h_{1,k}} + \Gamma_1 \wedge (Z_{1,k-1}Z_{1,k-1}^{\top} - \overline{h_{1,k}}) + \Gamma_2 \wedge (h_{1,k-1} - \overline{h_{1,k}}), \quad Z_k \sim \mathcal{N}(0, \partial_k)$$

where $\overline{h_{i,k}}$ is the unconditional covariance matrix of the multivariate standardized residuals $Z_{i,k}$, $\Gamma_1, \Gamma_2 \in \mathcal{R}_+$ are parameters constants in the equation. Note that the observations on the linear combination of the equations shows there is error corrections for conditional correlations.

### 3.8 Vector Parameters in Risk Neutral martingale $\mathcal{Q}$.

In the martingale $\mathcal{Q}$, consider the equations (3.5.7) and (3.5.8) such $\xi_{i,k}$ in equation (3.4.2) can be written as $\xi_{i,k} = \frac{1}{\sqrt{h_{i,k}}}\left(\sqrt{h_{i,k}^*}\xi_{i,k}^* + \Upsilon_{i,k}h_{i,k}^*\right)$, where

$\Upsilon_{i,k} = -\lambda_{i,k} - 0.5\mathcal{I}_{i\wedge i}$ and $\xi_{i,k}^* \sim \mathcal{N}(0, \mathcal{I})$ which Escobar et al. (2019) proposed that in vector form $\xi_k = \sqrt{h_k^*}\xi_k^* + b^*\sqrt{h_k^*}$. Substituting, we obtain the risk-neutral process given by

$y_{ii,k}^{\mathcal{Q}} = r_{ii} - 0.5h_{ii,k}^* + \sqrt{h_{ii,k}^*}\xi_k^*, \quad \xi_k^* \overset{\mathcal{Q}}{\sim} \mathcal{N}(0,1)$ here $r_{ii}$ is matrix for the risk-free rates and the conditional variance matrix is given by equation (3.5.8) with the risk-neutral dynamics. Escobar et al. (2019) stated that the return risk-neutral equation can therefore be represented in matrix vector form as $y_k^{\mathcal{Q}} = r\mathcal{I} + (1 - 0.5A \wedge A) \wedge h_k^* + A \wedge h_k^* \wedge \xi_k^*$,

It makes it obvious that the risk-free rate of return and predicted asset returns is comparable[1].

## 4 Data

In this section, we shall look at the empirical overview of the models, format of the data set used and its characteristics, simulations formats of the multivariate model.

### 4.1 Data Description

[1] A discounted martingale for stocks may be obtained by applying an analogous change of measure of the type $d\xi_k^* = d\xi_k - g_k\sqrt{h_k}dt$, provided that $A^{-1}AB + \frac{1}{2}A \odot A(D^{-1}) = G$, where G is a diagonal array containing elements $g_k$

The study uses high-frequency trading datasets from within a single trading day for options on DAX and SPX data of S&P 500 index downloaded from yahoo finance plat form using the TensorFlow. These datasets are credited with lots of money thus being appropriate for the training and testing the model. Ensuring the smart study, Christoffersen et al. (2013) explained why S&P 500 index do not have complications to value of options, Heston and Nandi (2000) stated that because of the wildcard's characteristics option under S&P index are simpler compared to S&P 100 option. The examination date of each index was chosen from 6/2023 to 9/2024 right after the impact of Covid-19. The evaluation in pre-processing of

DAX index between 6/2023 to 9/2024 was left out to the consideration of the reader. The SPX data index was found to have its components selected to pick companies across various industry to present across the US economy. The various pre-process were done and the statistical summary of the SPX data index was obtained as shown in the table (4.1) below.

**Table 4.1:** Table for statistical summary

| | Min. | 1st Qu. | Median |
|---|---|---|---|
| **Adj. Close** | 4115 | 4388 | 4556 |
| **Volume** | 1.64e+9 | 3.617e+9 | 3.843e+9 |
| | **Max.** | **3rd Qu.** | **Mean** |
| **Adj. Close** | 5321 | 5013 | 4674 |
| **Volume** | 8.219e+9 | 4.096e+9 | 3.924e+9 |

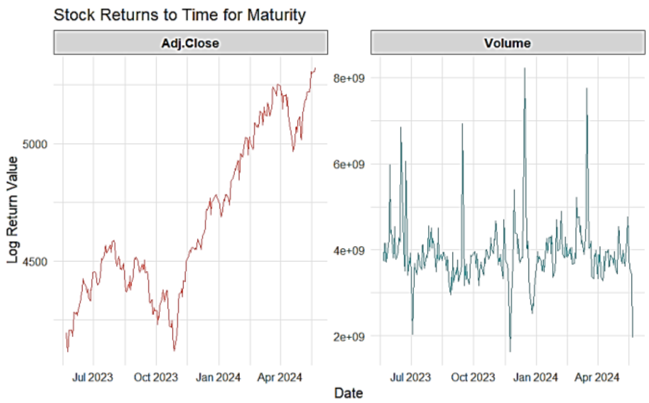
The empirical analysis includes previous S&P 500 data sets with an average yield of 4556 and lowest and maximum stock price values of 4115 and 5321, respectively. After examining the data's structure, we processed the dynamics of S&P 500 influence on DAX. As Bakshi et al. (1998) stated that the impact due to changes in data point's closing price may have on the corrected final prices of another asset within a portfolio are referred to as the "spillover effects of assets in a portfolio." Because it makes the links and inter dependencies between the numerous variables in the markets clear, this may affect volatility consistencies thus a fundamental in data pre-processing. The spillover effects of two data sets were examined and represented in the table (4.2)

**Table 4.2:** Table for spillover Effects of DAXI and SPX

| Index | DAX Spillovers | S&P 500 Spillovers |
|---|---|---|
| 0 | NaN | NaN |
| 1 | -13.979980 | -47.049805 |
| 2 | -43.839844 | -30.339844 |
| 3 | 22.750000 | 36.039551 |
| 4 | 0.450195 | 54.170410 |

The question to ask is this enough information to justify the variations of spillover effects. To be gain broad understanding in the examination of the corporations within the data to be used, we need a plot of the volatility and returns in the target column of volume and Adj.close respectively.

**Figure 4.1:** Figure shows how stock prices of S&P 500 data used vary per day and average log- returns.



The graphs shows that stock return have stock market prices which experience the clustering volatility. The volume graph shows the variation of average returns while graph of Adj.close shows how the variations of prices. Since the volatility is seen to evolve independently, this allows option pricing across multiple underlying assets to be analyzed.

### 4.2 Simulation analysis

In this section, our main goal is to build frameworks that can correctly forecast stock price growth without neglecting time-varying volatility into account. According to Christoffersen et al. (2013), the GARCH(1,1) model has been shown to be a sound empirical model with regard to volatility analysis. For illustration, we use the equation (3.7.3) for MuGARCH(1,1) model and equation (3.2.1) in this section. We simulate both models using the Monte Carlo.

**4.2.1 Simulation in Risk- Neutral GARCH(1,1) Model.** Given all the information in a $\mathcal{F}_k$ filed for reproducing stock price, we can define the one-step variance function, consisting the conditional variance dynamics of the stock price process in equation (3.2.1). Model parameters $\omega$, $\beta_1$, $\alpha_1$ and a risk premium $\lambda$ were assumed for 50 number of simulations in three intervals as shown in table (4.3).

**Table 4.3:** Table showing assumed parameter values in Risk Neutral GARCH model

| Parameters | 1st Interval | 2nd Interval | 3rd Interval |
|---|---|---|---|
| $\omega$ | 0 | 5e-5 | 0.4 |
| $\alpha$ | 8.887e-7 | 1.23e-6 | 9.1e-8 |
| $\beta$ | 0.751 | 0.651 | 0.800 |
| $\lambda$ | 0.574 | 0.685 | 0.712 |

By evaluating the standard deviations allows us to capture the instantaneous influence of volatile markets on turbulence. We also made daily simulations of yearly interest rate and volatility, the annualization was done based on 256 days.

**4.2.2 Simulations in MuGARCH model.**

As in Duan (1995), this section we assumed that the risk-free interest rate r is zero throughout. In addition, the assumption of forecasts on past volatility and returns as present/ constant volatility was violated in stochastic market models where volatility is random process. Monte Carlo simulations in conjunction with its variance reduction methods as indicated in the literature of Silvennoinen and Terasvirta (2008) and Bollerslev (1990) who proposed a format to parameterize the multivariate model with conditional variances, and covariance but constant and dynamical correlations. The Dynamical conditional covariance model was chosen because it assigns fewer parameters to stand in for the volatility and formulation of returns. Firmly putting into account, the estimate and dynamic correlations with less computational complexity, the MuGARCH model is computationally easy relative to other GARCH models. For this case, we explained the algorithm of simulation under this model as earlier said in section (3.7). In continuation, we need to develop an update for $h_k$ in each time-step variation, we consider rebuilding the dynamic correlation matrix in equation (3.7.3). The unrestricted parameters under variance $h_k$ and $Z_k$ ca n be obtained from the equation below.

$$\overline{h^{\mathcal{P}}}_{i,k} = \frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{n} Z_{i,k-1} \wedge Z_{i,k-1}^{\top} \quad n=2 \qquad (4.2.1)$$

for bivariate models

$$Z_k = \left(\mathbf{diag}(\zeta_{i,k}^2)\right)^{-\frac{1}{2}} y_k^{\mathcal{P}} \qquad (4.2.2)$$

Keeping in mind that we are dealing with bivariate we set $i$ = 2 observations thought the entire time line stepping. If we assume the random parameters from (2.2.6) and (3.7.3) as show in the table (4.4) below.

**Table 4.4:** Table showing the assumed parameters in MuGARCH

| **Parameter** | $\omega_1, \omega_2$ | $\alpha_1, \alpha_2$ | $\beta_1, \beta_2$ | $\Gamma_1, \Gamma_2$ |
|---|---|---|---|---|
| **Value** | 0.02, 0.025 | 0.08, 0.045 | 0.89, 0.94 | 0.04, 0.88 |

Considering all the variable parameters, MuGARCH model is imported to have different paths of simulations through the described algorithm. Silvennoinen and Terasvirta (2008) and Laurent (2021) proposed the importance of using a iterative technique of updating the covariance matrix instead of predicting the covariance matrix up to a time iteration, k and formulating the normalized path vector products with the covariance matrix. In each iteration, there is no considerable impact between the covariance vector values of the current and incoming days. For both multivariate and univariate, first step on the algorithm is to predetermine the parameters in table (4.4). As indicated before the estimations strongly depend on the normalized residuals $Z_k$ in equation (4.2.2). The choice for the distribution depends on the data but in this paper, we consider the multivariate normal distribution for the residuals. Considering the normalized error $Z_k$ with time iteration $k$ = 1, 2, 3, …., $t$. [1]Here $t$ = 1, …, $T$ is the time period used to estimate the model. Laurent (2021) used the linear transformation rule of variables and the algorithmic steps described above to give the likelihood function $\ln L(\mathrm{Y})$ for $y_k = \left(\mathbf{diag}(\zeta_{i,k}^2)\right)^{\frac{1}{2}} Z_k$ as [2]

## 5 Results and Analysis

In this section we present the results and their discussions obtained from the simulations made in the Risk-Neutral model, MuGARCH(1,1) model.

### 5.1 Results and Discussions from Simulations under Risk Neutral model

The 1000 simulations were carried out but two are presented in the figures (5.1a) and (5.1b) for time runs. Despite the fact that all the variation in returns show both high and low values implies that predictions can be made by the model to fit any data.

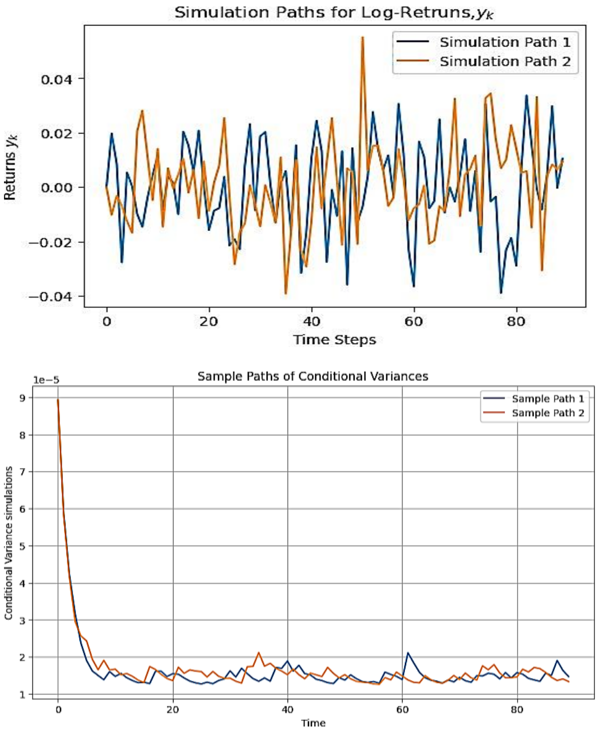
The realizations of the innovation series generate different beginning points even when given the similar starting variance. This may be due to the random selected distribution of $\xi$ from Gaussian with mean value 0 and variance, 1.

**Figure 5.1:** Figures showing Comparison of Simulated Returns and Volatility

[1] The joint distribution is then given $\prod_{k=1}^{T} \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2} Z_k^T Z_k\right\}$ since $\mathbb{E}[z_k] = 0$ and $\mathbb{E}[z_k z_k^T] = I$.

[2] There is indeterminably log-likelihood function, thus the model can be allowed to take on the second formulation. This paper focuses on this formulation.

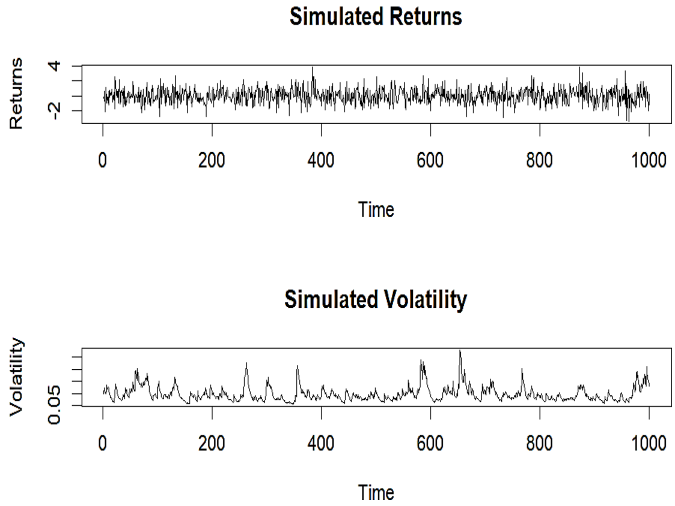
$$-\frac{1}{2}\sum_{k=1}^{T}\left( n\ln(2\pi) + \ln\left( |\,\mathbf{diag}\left(\zeta_k^2\right)|^{-\frac{1}{2}} \right) + \ln(|\,\partial_t\,|) + y_k^T \left(\mathbf{diag}\left(\zeta_k^2\right)\right)^{-\frac{1}{2}} \partial_k^{-1} \left(\mathbf{diag}\left(\zeta_k^2\right)\right)^{-\frac{1}{2}} y_k \right)$$




In figure (5.1b), however the conditional variance is seen to decrease from $9e^{-5}$ to approximate value of $1.6e^{-5}$. Both simulations take a decreasing trend however both paths tend to maintain a variance of $1.6e^{-5}$ beyond time iterations of > 10. The decreasing conditional variance provides a good sense in which the parameters values were assumed, the simulations decrease the trend asymptotically on $1e^{-5}$ due to the difference positive correlation and may be the positive value of the risk premium, $+\lambda$.

Having looked at the log-returns in the risk neutral measures we can now find the randomness in the simulation market of pricing options. The Risk neutral GARCH(1,1) don't account for structure reliance and therefore it is absolutely pure from normalized errors, this implies that as the simulations increase the unconditional variance $\rightarrow \infty$ which is violated by the parameter assumed. However from equation (4.2.2), the *i.i.d* residuals follows a univariate standard normal distribution thus having absolutely no correlation format. Lets focus on the multiple simulations yielding returns and volatility of the interval parameters assumed before in table (4.3). The volatility trends of all the interval cases shown in figure (5.2) vary almost the same in the simulations ranging from 400 <, and < 800 and exhibit clustering for all the simulations.

**Figure 5.2:** Interval's volatility and returns evaluation




We simulated the model depending on the random vectors errors in table (5.1). The values of the standard deviations and payoff for European call from the simulated parameters was obtained and recorded.

Keeping the random vector errors constant, we used the parameters set by Duan (2000) to obtain almost the same approximate values as presented in table (5.2a) and table (5.2b)

The initialization of current stock price $S(0) = 51$, standard deviation $SD(1) = 0.2$, the daily risk-free rate and standard deviations were calculated per annum. Although the dynamics of this risk neutral model are not realistic with suitable choice of parameters, the model approximates roughly the same values for the standard Monte Carlo simulations in Duan (2000).

**Table 5.1:** Table showing Terminal Stock prices

| Error Simulations | | | |
|---|---|---|---|
| Error 1 | Error 2 | Standard Deviation | Stock Prices |
| 0.7809 | 2.3587 | 90.35161 | 46.95111 |
| 2.0049 | 1.6775 | 90.13422 | 49.10413 |
| 2.1353 | 1.4168 | 90.39232 | 45.07714 |
| 1.9637 | -0.4495 | 90.56557 | 47.84593 |
| 1.3512 | 0.2180 | 90.15646 | 47.84593 |
| 0.7809 | 2.3587 | 90.35161 | 46.95111 |
| 2.0049 | 1.6775 | 90.13422 | 49.10413 |
| 2.1353 | 1.4168 | 90.39232 | 45.07714 |
| 1.6646 | -0.4495 | 90.56557 | 47.84593 |
| 1.3512 | 0.2180 | 90.15646 | 47.84593 |

In the both table (5.2a) and table (5.2b), the values of the standard deviation obtained by the risk neutral model. The parameter values for the two cases of intervals are presented and found to be the same for this model. More considerations can be taken to predict returns in financial data index and price derivatives using the model. We can use the obtained model to price options, the valuation of prices is represented in the table (5.3).

**Table 5.2a:** Table showing comparisons of simulations

| S(0) | SD(1) | S(1) | Call 1 | S(1) |
|---|---|---|---|---|
| 51 | 0.200 | 50.572 | 1.012 | 50.57189 |
| 51 | 0.200 | 50.713 | 1.022 | 50.71296 |
| 51 | 0.200 | 51.224 | 1.271 | 51.22406 |
| 51 | 0.200 | 51.238 | 0.921 | 51.23806 |
| 51 | 0.200 | 51.294 | 1.208 | 51.29403 |
| 51 | 0.200 | 50.448 | 1.881 | 50.44811 |
| 51 | 0.200 | 51.202 | 1.243 | 51.20197 |
| 51 | 0.200 | 49.925 | 0.000 | 49.92468 |
| 51 | 0.200 | 50.875 | 0.151 | 50.87472 |
| 51 | 0.200 | 51.169 | 1.371 | 51.16950 |

**Table 5.2b:** Table showing comparisons of simulations

| SD(2) | S(2) | SD(2) | Call 2 | S(2) |
|---|---|---|---|---|
| 0.215 | 51.012 | 0.342226 | 0.571734 | 50.561 |
| 0.207 | 51.022 | 0.294625 | 0.712768 | 50.702 |
| 0.190 | 51.271 | 0.123271 | 1.223723 | 51.274 |
| 0.190 | 50.921 | 0.118602 | 1.237717 | 51.248 |
| 0.190 | 51.208 | 0.099945 | 1.293677 | 51.294 |
| 0.222 | 51.881 | 0.384103 | 0.447984 | 50.448 |
| 0.191 | 51.243 | 0.130641 | 1.201641 | 51.201 |
| 0.261 | 48.918 | 0.562326 | 0.000000 | 49.964 |
| 0.200 | 50.151 | 0.240208 | 0.874477 | 50.844 |
| 0.191 | 51.371 | 0.141481 | 1.169178 | 51.199 |

By setting to first interval of parameters in table (4.4),the valuation for each time to maturity, the effect on options can be attached to the valuation of strike prices. This is because prices of higher strike prices have Option prices ranging from [0.3266, 10.4983] respective from lowest to highest maturity times. The differences in strike prices show how intuitively option prices are to changes in the prices of the underlying asset. The longer time value component led to higher option prices; this is due to the certainties on in the money market price situations. The option prices for the second and third interval set of parameters are given and represented in the tables (5.4) and (5.5). In all the obtained results for the three different intervals of parameters, the price of a European call options exhibit similar trends that is decrease as the strike prices increase. This is a widely recognized characteristic of option prices that is irrespective of the model used. It is important for an investor to know such variations for instance when the strike price exceed the market price, the call options are said to be in the market with an intrinsic value while when the strike exceeds the market value the option derivatives are said to be out the money with no intrinsic value.

**Table 5.3:** Table showing Option Prices for Different Strike Prices and Maturities

| TIME TO MATURITY | | | | | | |
|---|---|---|---|---|---|---|
| **Strike Price** | 30 | 60 | 90 | 120 | 150 | 180 |
| 90 | 20.6401 | 21.4731 | 22.714 | 24.5966 | 27.1989 | 30.6185 |
| 92.2222 | 16.4001 | 17.4437 | 19.0335 | 21.2498 | 24.0689 | 27.6532 |
| 94.4444 | 12.3055 | 13.6293 | 15.6227 | 18.1522 | 21.1433 | 24.8667 |
| 96.6667 | 8.4158 | 10.1528 | 12.5477 | 15.3199 | 18.4399 | 22.2644 |
| 98.8889 | 4.8828 | 7.1861 | 9.8502 | 12.7686 | 15.9604 | 19.8476 |
| 101.1111 | 2.581 | 4.8828 | 7.5629 | 10.5053 | 13.7125 | 17.6136 |
| 103.3333 | 1.6263 | 3.2735 | 5.6784 | 8.5263 | 11.6921 | 15.5652 |
| 105.5556 | 1.0045 | 2.2372 | 4.1823 | 6.8288 | 9.8921 | 13.7018 |
| 107.7778 | 0.5883 | 1.5605 | 3.0372 | 5.3996 | 8.3076 | 12.0156 |
| 110 | 0.3266 | 1.0862 | 2.1862 | 4.2105 | 6.9209 | 10.4983 |

**Table 5.4:** Table showing Option Prices for Different Strike Prices and Maturities for second interval of parameters.

| TIME TO MATURITY | | | | | | |
|---|---|---|---|---|---|---|
| Strike Price | 30 | 60 | 90 | 120 | 150 | 180 |
| 90 | 20.6664 | 21.6331 | 22.7626 | 24.0592 | 25.5664 | 27.272 |
| 92.2222 | 16.4264 | 17.5735 | 18.8543 | 20.2976 | 21.9467 | 23.7755 |
| 94.4444 | 12.3318 | 13.6706 | 15.1345 | 16.7434 | 18.5337 | 20.4754 |
| 96.6667 | 8.4407 | 9.9963 | 11.6852 | 13.4646 | 15.3707 | 17.4019 |
| 98.8889 | 4.8436 | 6.7318 | 8.6263 | 10.5281 | 12.4952 | 14.5837 |
| 101.1111 | 2.509 | 4.2804 | 6.1151 | 8.0035 | 9.9537 | 12.0424 |
| 103.3333 | 1.6263 | 2.9398 | 4.3143 | 5.9676 | 7.7809 | 9.7976 |

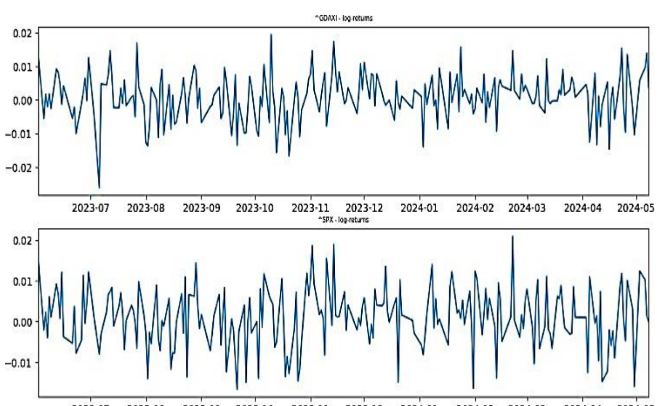
**Table 5.5:** Table showing Option Prices for Different Strike Prices and Maturities in third interval of parameters

| TIME TO MATURITY | | | | | | |
|---|---|---|---|---|---|---|
| Strike Price | 30 | 60 | 90 | 120 | 150 | 180 |
| 90 | 10.6809 | 11.5608 | 12.4332 | 13.2623 | 14.0479 | 14.7949 |
| 92.2222 | 8.6496 | 9.6948 | 10.6634 | 11.5562 | 12.3879 | 13.1704 |
| 94.4444 | 6.7638 | 7.9729 | 9.0251 | 9.9704 | 10.8391 | 11.6495 |
| 96.6667 | 6.7638 | 7.9729 | 9.0251 | 9.9704 | 10.8391 | 11.6495 |
| 98.8889 | 5.0815 | 6.4213 | 7.5338 | 8.5155 | 9.4091 | 10.238 |
| 101.1111 | 3.6515 | 5.0595 | 6.2004 | 7.1983 | 8.1028 | 8.9397 |
| 103.3333 | 2.5009 | 3.8967 | 5.0298 | 6.022 | 6.9223 | 7.7561 |

### 5.3 Results and Discussions of MuGARCH (1,1) model

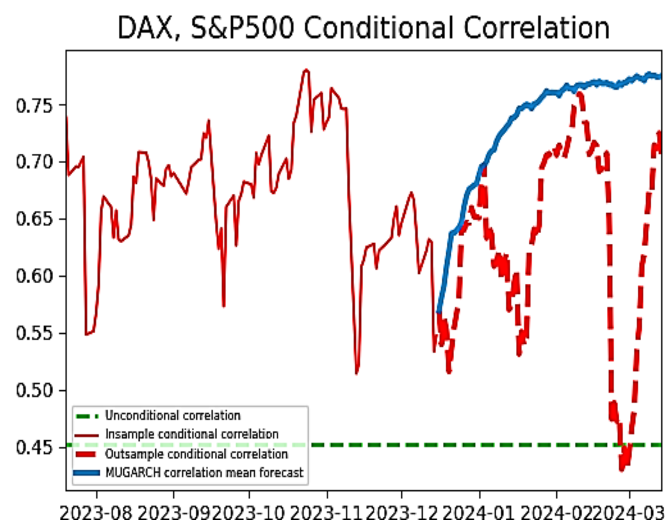
The pre-processing of the datasets DAXI and SPX index were followed from section (4.1), for which the Dynamic Conditional Covariance MuGARCH model produces specified trends of the datasets. Notably recalling that the model specifications are for normalized distributions, proposition can be taken for student t-distributions which could nicely work as well in the same trajectories of the simulations. For these datasets, the examinations were taken on correlations using the matrix vector equation (3.7.3) and the comparison graph was plotted as shown in figure (5.3) below.

**Figure (5.3a)** A graph showing Log-returns




For each time-series, the characteristic volatility clusters are apparent. The link connecting the two indicators seems to have decreased after the epidemic started. After then, correlation ap- pears to be cyclical. Overall, the trend resembles an Ornstein-Uhlenbeck processes discretized. Our structure's error correction approach in equation (3.7.3) should be able to account for this behavior. Now the focus should be on whether the model can be used to forecast correlations see figure (5.3b) , the graph show outcomes of equation (3.6.1), $h_{ii,k}$ is conditional correlation represented by the red curve, cyan colored curve represents the conditional correlations iterated at some time, $k$ and green line represents the unconditional correlations precisely better.

**Figure (5.3b):** A graph of Conditional Correlations




Observations show that the conditional correlations are often greater than the unconditional correlations, there are two sharp syncline correlations during the month of August 2023 and late April 2024 which affected the two-market index. This doesn't necessary mean that the DAXI index has an effect on to SPX and vice versa but further analysis can be taken for proper portfolio management. The conditional correlations show how one may hedge a DAXI option on SPX option index. For empirical asymmetric methods, it might be interesting to be able to predict correlations with reliability.

Correlations could be an intriguing substitute for price changes, which are the usual method used in such tactics. Thus far, our focus has been on comprehending the discretized matrix form of MuGARCH' s conditional correlation predictions (see 5.3b). By doing so, we hope to be able to apply the model for option pricing by gaining a grasp of the correlations' simulated running movements. The predicted correlations of the model move along with the running correlations see figure (5.3d) below up to the time to maturity thus we have created a reflection set in MuGARCH that will allow us compare the conditional variances and correlations notably for special and dynamical assumptions on the parameter models

**Figure (5.3d):** A graph of Running Correlation

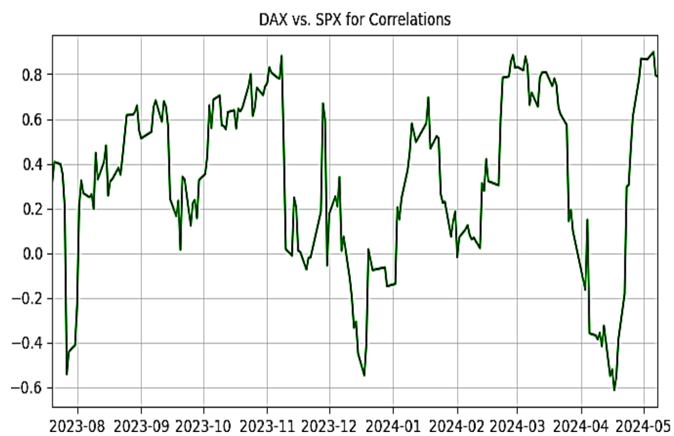
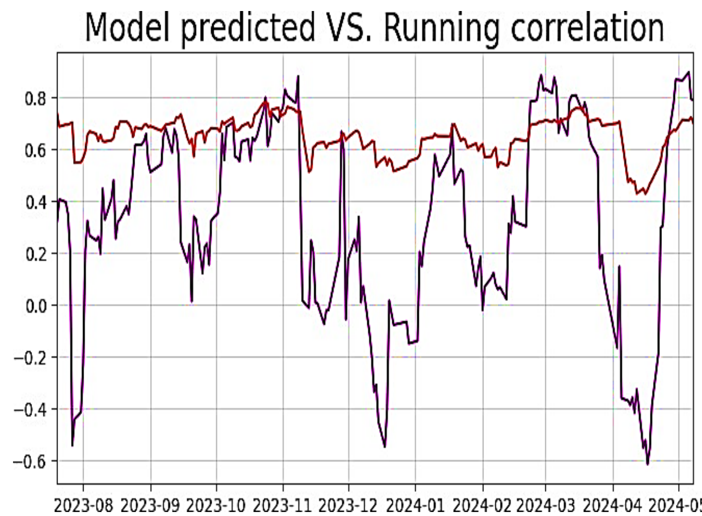



**Figure (5.3d):** Predicated and Running Correlation




The prediction line in figure (5.3d) fits for fewer estimates of the running correlations for time variants. Further analysis, training and modifications may be done for proper fitting the model.

## 6. Conclusions and Perspectives

For the univariate format, the initial parameters were needed for predictions in the option markets and properly fitted to exhibit the relationship equation of decrease in strike price increases in option prices. The datasets were used from the 1st day of June 2023 and the examinations were done on SPX data index and the DAXI was assumed to have the same trend with all the volatility clustering. Gaussian distributions of the Dynamic condition correlations version of multivariate GARCH were considerations and it's modelling procedures are well stated. Although the methodology in pricing derivatives has not been documented, the paper shows how the running conditional correlations is determined with the fit of the model. The aim of this paper was to model and simulate paths in discrete-time frameworks of GARCH, focusing on univariate and multivariate the assumptions on parameters in both models and standard datasets for the models were based on the considerations of the author. The paper also shows the precise derivations of univariate and multivariate GARCH models in both martingale states. In this, however, we focus on the stochastic discount factor, locally risk-neutral valuation, and the conditional Esscher transforms as powerful tools for deriving these models. The normality assumptions were presented as an absolute necessity for all the residual information in the market. The moment-generating functions and cumulant-generating functions for the respective univariate and multivariate GARCH were presented and discussed to an absolute unit of scale. The time variant conditional correlation and covariance matrix formulas were presented in their discretized forms, and it was found that there are error corrections in the formula. While it has been an objective to derive these formulas, computational simulations were also presented, and models prescribed the trend of clustering. Option assessment research remains subject to computational limitations because of the size of the data sets and the complexity of the processes required by the data's construction, even when advances in computing power constitute the evaluation of data- rich and complex optimization issues become more feasible with graphing. The major discovery in this paper is that the MuGARCH (1,1) was a good estimator to a time variate conditional correlation.

Fewer models were considered to simply reduce the content of the paper for the interest of time. Therefore, the need to address more models will provide much more insights into investor choices in index comparisons. In addition, this paper gives parameter and simulation insights but one may practice improving GARCH equations for up-coming problems.

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In person to my parents and friends irrespective of hierarchy and sounding more openly I am thankful to you all. I humbly submit my ton of appreciation to all academic personnel who published their work as an open-end source. Since the bibliography is meant to be a list of sources for the current text, there are various names that articulately influenced my work and are intrinsically referenced or mentioned to in the extremities of this paper.

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